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Liouvillian first integrals of quadratic–linear polynomial differential systems

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ABSTRACT

For a large class of quadratic–linear polynomial differential systems with a unique singular point at the origin having non-zero eigenvalues, we classify the ones which have a Liouvillian first integral, and we provide the explicit expression of them.

1. Introduction

For planar differential systems the notion of integrability is based on the existence of a first integral. For such systems the existence of a first integral determines completely its phase portrait. Then a natural question arises: *Given a system of ordinary differential equations in $\mathbb{R}^2$ depending on parameters, how to recognize the values of such parameters for which the system has a first integral?*

In particular the planar integrable systems which are not Hamiltonian, i.e. the systems in $\mathbb{R}^2$ that cannot be written as $x' = -\partial H/\partial y$, $y' = \partial H/\partial x$ for some function $H : \mathbb{R}^2 \to \mathbb{R}$ of class C^2, are in general very difficult to detect. Here the prime denotes derivative with respect to the independent variable t.

The first step to detect those first integrals in different classes of functions, namely polynomial, rational, elementary or Liouvillian, is to determine the algebraic invariant curves (i.e., the so-called Darboux polynomials).

Let P and Q be two real polynomials in the variables x and y, then the system

$$x' = P(x, y), \qquad y' = Q(x, y), \tag{1}$$

is a *quadratic polynomial differential system* if the maximum of the degrees of the polynomials P and Q is two.

Quadratic polynomial differential systems have been investigated for many authors, and more than one thousand papers have been published about these systems (see for instance [14] and [16]), but the problem of classifying all the integrable quadratic polynomial differential systems remains open.

Let $U \subset \mathbb{R}^2$ be an open set. We say that the non-constant function $H : U \to \mathbb{R}$ is a first integral of the polynomial vector field X on U, if $H(x(t), y(t)) =$ constant for all values of t for which the solution $(x(t), y(t))$ of X is defined on U. Clearly H is a first integral of X on U if and only if $XH = 0$ on U.

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The study of the Liouvillian first integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville, see for details again [15]. A *Liouvillian first integral* is a first integral H which is a Liouvillian function, that is, roughly speaking which can be obtained "by quadratures" of elementary functions. For a precise definition see [15].

As far as we know the Liouvillian first integral of some multi-parameter family of planar polynomial differential systems has only been completed classified for the planar Lotka–Volterra system of degree 2, see [3,10–13].

It was proved in [9] (see Proposition 3), that any quadratic–linear differential system

$$x' = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \qquad y' = A + Bx + Cy, \tag{2}$$

with $A^2 + B^2 + C^2 \neq 0$ having a unique finite singular point with non-zero eigenvalues, through a linear change of variables and a rescaling of the time can be written into the form

$$x' = P(x, y) = bx + cy + dx^2 + exy + fy^2, \qquad y' = Q(x, y), \tag{3}$$

where $P(x, y) \neq bx + cy$ and $Q(x, y)$ is either x or y. Moreover,

(S1) if $Q(x, y) = y$, then $d = 0$, $b \neq 0$ and $e^2 + f^2 \neq 0$;
(S2) if $Q(x, y) = x$, then $f = 0$, $c \neq 0$ and $d^2 + e^2 \neq 0$.

We do not consider in (3) the case $Q(x, y) = 0$ because the possible singular points have always a Jacobian matrix with zero eigenvalues. We do not consider in (3) the case $Q(x, y) = 1$ since it has no singular points. When $Q(x, y) = y$ then $d = 0$ and $b \neq 0$. Indeed, the singular points of (3) satisfy in this case $y = 0$ and $P(x, y) = x(b + dx) = 0$. Therefore, since we want the origin to be the unique singular point we must have $bd = 0$ with $b^2 + d^2 \neq 0$. If $d \neq 0$ then $b = 0$ and in this case the Jacobian matrix at the origin has a zero eigenvalue, so we do not consider this case. Therefore we must have $d = 0$ and $b \neq 0$. Furthermore, since $P(x, y)$ must be quadratic and $d = 0$, we must have $e^2 + f^2 \neq 0$. Proceeding in a similar way when $Q(x, y) = x$ in order that it has only the origin as a singular point with Jacobian having non-zero eigenvalues, we must have $f = 0$ and $c \neq 0$. Furthermore, in order that $P(x, y)$ be quadratic we must have $d^2 + e^2 \neq 0$.

Our first result is the following.

Theorem 1. *System* (3) *satisfying* (S1) *is integrable.*

(a) *If $e \neq 0$, then the first integral is*

$$H = \exp(-ey)y^{-b}\big(ex + fy + \exp(ey)\big(ce + (1 - b)f\big)yEI_b(ey)\big), \tag{4}$$

where $EI_b(x)$ is the exponential integral function

$$EI_b(x) = \int_1^{\infty} \frac{\exp(-xt)}{t^b}\,dt = x^{b-1}\Gamma(1 - b, x) \quad \text{for any } b \in \mathbb{R},$$

where Γ is the incomplete gamma function, for more details see [1].

(b) *If $e = 0$ and $(b - 1)(b - 2) \neq 0$, then the first integral is*

$$H = y^{-b}\big((b - 1)(b - 2)x + y\big((b - 2)c + (b - 1)fy\big)\big). \tag{5}$$

(c) *If $e = 0$ and $b = 1$, then the first integral is*

$$H = \frac{x}{y} - fy - c\log y. \tag{6}$$

(d) *If $e = 0$ and $b = 2$, then the first integral is*

$$H = \frac{x + cy}{y^2} - f\log y. \tag{7}$$

The proof of Theorem 1 is given in Section 3.

Our second result is the following.

Theorem 2. *The unique Liouvillian first integrals $H = H(x, y)$ of system* (3) *satisfying* (S2) *are*:

(a) $H = (c + ex)^{c/e^2}\exp(y^2/2 - x/e)$ *if $d = b = 0$;*
(b) $H = \exp(-2dy)(2cdy + c + 2d^2x^2)$ *if $b = e = 0$.*