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Spike-layer solutions to singularly perturbed semilinear systems of coupled Schrödinger equations

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ABSTRACT

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \leqslant 3)$, we are concerned with the interaction and the configuration of spikes in a double condensate by analyzing the least energy solutions of the following two couple Schrödinger equations in Ω

$$\begin{cases} -\varepsilon^2 \Delta u + u = \mu_1 u^3 + \beta u v^2, \\ -\varepsilon^2 \Delta v + v = \mu_2 v^3 + \beta u^2 v, \\ u > 0, \quad v > 0, \end{cases} \tag{S_ε}$$

where μ_1, μ_2 are positive constants. We prove that under Neumann or Dirichlet boundary conditions, for any $\varepsilon > 0$, when $-\infty < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, system (S_ε) has a least energy solution $(u_\varepsilon, v_\varepsilon)$ and when $\min\{\mu_1, \mu_2\} < \beta < \max\{\mu_1, \mu_2\}$, system (S_ε) has no solution. Suppose $P_\varepsilon, Q_\varepsilon$ are the local maximum points of $u_\varepsilon, v_\varepsilon$ respectively. Then under Neumann boundary conditions, as ε small enough, both of $P_\varepsilon, Q_\varepsilon$ locate on the boundary of Ω. Furthermore, when $\beta \geqslant 0$, $\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to 0$ as $\varepsilon \to 0$ and for $N = 2$ and $N = 3$, $P_\varepsilon, Q_\varepsilon$ converge to the same point on the boundary which is the maximum point of mean curvature of the boundary. However, when $\beta < 0$, $\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to \infty$ as $\varepsilon \to 0$ and suppose $P_\varepsilon \to P$ and $Q_\varepsilon \to Q$, then for $N = 2$ and $N = 3$, P, Q must be the maximum points of the mean curvature on the boundary and P, Q might be a same point if the mean curvature of the boundary has only one maximum point. Under Dirichlet boundary conditions, we can prove that as long as the least energy solution (S_ε) exists, the same asymptotic behavior of the least energy solution $(u_\varepsilon, v_\varepsilon)$ holds as described in Lin and Wei (2005) [10] for $\beta > 0$ or for $\beta < 0$, thus our results are an extension of the results in Lin and Wei (2005) [10].

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \leqslant 3)$, we are concerned with the following two couple Schrödinger equations in $H^1(\Omega) \times H^1(\Omega)$ (or $H_0^1(\Omega) \times H_0^1(\Omega)$)

$$\begin{cases} -\varepsilon^2 \Delta u + u = \mu_1 u^3 + \beta u v^2, \\ -\varepsilon^2 \Delta v + v = \mu_2 v^3 + \beta u^2 v, \\ u > 0, \quad v > 0, \end{cases} \tag{S_ε}$$

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under Neumann (or Dirichlet) boundary conditions, where μ_1, μ_2 are positive constants and without loss of generality we assume $\mu_1 \leqslant \mu_2$, $\beta \in \mathbb{R}^N$, $\varepsilon > 0$ is the parameter.

For $\Omega = \mathbb{R}^N$ and $\varepsilon = 1$, (S_ε) leads to investigate the following problem in $\mathbb{R}^N$

$$\begin{cases} -\Delta u + u = \mu_1 u^3 + \beta u v^2, \\ -\Delta v + v = \mu_2 v^3 + \beta u^2 v, \\ u > 0, \quad v > 0, \\ u(x) \to 0, \quad v(x) \to 0 \quad \text{as } |x| \to \infty. \end{cases} \tag{1.1}$$

Problem (1.1) arises in the Hartree–Fock theory for a double condensate i.e. a binary mixture of Bose–Einstein condensate in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [9]). Physically, u and v are the corresponding condensate amplitudes, μ_j and β are the intraspecies and interspecies scattering lengths. The sign of the scattering length β determines whether the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive or attractive. When $\beta > 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are repulsive. In contrast, when $\beta < 0$, the interactions of states $|1\rangle$ and $|2\rangle$ are attractive.

Recently, B. Sirakov [13] discussed the whole $\beta \in \mathbb{R}$ and analyzed for which β problem (1.1) assures a least energy solution and for which β problem (1.1) has no least energy solution.

When the domain in (1.1) is replaced by a symmetric domain (possibly unbounded), T. Bartsch, N. Dancer and Z.Q. Wang [4] investigated the local and global bifurcation in terms of the parameter β which provides a-priori bounds of solution branches.

We also refer the readers to Antonio Ambrosetti and Eduardo Colorado [1,2] for the bound states of Schrödinger equations and T. Bartsch, Z.Q. Wang and J. Wei [5], T. Lin and J. Wei [11,12], J. Wei and T. Weth [14,15], L.A. Maia, E. Nontefusco and B. Pellacci [3] for the bound states of Schrödinger systems.

In particular, T. Lin and J. Wei [10] considered (S_ε) under Dirichlet boundary conditions, they obtained the existence of the least energy solution to (S_ε) by minimizing the certain Nehari manifold for $-\infty < \beta < \beta_0$ and also discussed the asymptotic behavior as ε goes to zero, where $0 < \beta_0 < \sqrt{\mu_1 \mu_2}$ is a constant depending only on n. More precisely, they pointed out that when $\beta < 0$, the maximum points of the two components of the least energy solution to (S_ε) approach different points as $\varepsilon \to 0$ whereas when $0 < \beta < \beta_0$, the maximum points of the two components of the least energy solution to (S_ε) go together as $\varepsilon \to 0$.

In present paper, we firstly consider (S_ε) under the Neumann boundary conditions, namely we consider the following problem in $H^1(\Omega) \times H^1(\Omega)$

$$\begin{cases} -\varepsilon^2 \Delta u + u = \mu_1 u^3 + \beta u v^2, \\ -\varepsilon^2 \Delta v + v = \mu_2 v^3 + \beta u^2 v, \\ u > 0, \quad v > 0, \\ \dfrac{\partial u}{\partial n} = 0, \quad \dfrac{\partial v}{\partial n} = 0, \quad \text{on } \partial\Omega, \end{cases} \tag{S_ε^1}$$

where $\frac{\partial}{\partial n}$ denotes the external normal derivative on the boundary.

A solution (u, v) of (S_ε^1) which has a zero component ($u \equiv 0$ or $v \equiv 0$) will be called a standard solution. $(0, 0)$ is referred as the trivial solution of (S_ε^1). We are concerned on the nonstandard solutions of (S_ε^1) and also their asymptotic behavior as ε approaches zero.

The energy functional corresponding to (S_ε^1) is as follows:

$$J_\varepsilon(u, v) := \frac{1}{2} \int_\Omega \left[\varepsilon^2 |\nabla u|^2 + u^2 + \varepsilon^2 |\nabla v|^2 + v^2\right] dx - \frac{1}{4} \int_\Omega \left(\mu_1 u^4 + \mu_2 v^4 + 2\beta u^2 v^2\right) dx, \tag{1.2}$$

for every $(u, v) \in H^1(\Omega) \times H^1(\Omega)$.

As in [10], we consider the set

$$\mathcal{N}(\varepsilon, \Omega) =: \left\{ (u, v) \in H^1(\Omega) \times H^1(\Omega),\ u \gneqq 0,\ v \gneqq 0\colon \begin{matrix} \int_\Omega [\varepsilon^2 |\nabla u|^2 + u^2] = \int_\Omega [\mu_1 u^4 + \beta u^2 v^2] \\ \int_\Omega [\varepsilon^2 |\nabla v|^2 + v^2] = \int_\Omega [\mu_2 v^4 + \beta u^2 v^2] \end{matrix} \right\}$$

and let

$$c_\varepsilon = \inf_{(u,v) \in \mathcal{N}(\varepsilon, \Omega)} J_\varepsilon(u, v).$$

Our first results deal with the existence of least energy solutions of (S_ε^1) which achieve c_ε.

Theorem 1.1. *For any $\varepsilon > 0$, if $-\infty < \beta < \min\{\mu_1, \mu_2\}$ or $\beta > \max\{\mu_1, \mu_2\}$, there exists a least energy solution $(u_\varepsilon, v_\varepsilon)$ to system (S_ε^1) which achieves c_ε. If $\min\{\mu_1, \mu_2\} < \beta < \max\{\mu_1, \mu_2\}$, (S_ε^1) has no solution. In fact, suppose ω_ε is a least energy solution*