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Analytic mappings between noncommutative pencil balls

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ABSTRACT

In this paper, we analyze problems involving matrix variables for which we use a noncommutative algebra setting. To be more specific, we use a class of functions (called NC analytic functions) defined by power series in noncommuting variables and evaluate these functions on sets of matrices of all dimensions; we call such situations dimension-free. These types of functions have recently been used in the study of dimension-free linear system engineering problems (Helton et al. (2009) [10], de Oliviera et al. (2009) [8]). In the earlier paper (Helton et al. (2009) [9]) we characterized NC analytic maps that send dimension-free matrix balls to dimension-free matrix balls and carry the boundary to the boundary; such maps we call "NC ball maps". In this paper we turn to a more general dimension-free ball $\mathcal{B}_L$, called a "pencil ball", associated with a homogeneous linear pencil

$$L(x) := A_1x_1 + \cdots + A_gx_g, \quad A_j \in \mathbb{C}^{d' \times d}.$$

For $X = \operatorname{col}(X_1, \dots, X_g) \in (\mathbb{C}^{n\times n})^g$, define $L(X) := \sum A_j \otimes X_j$ and let

$$\mathcal{B}_L := \left(\left\{X \in \left(\mathbb{C}^{n\times n}\right)^g \colon \left\|L(X)\right\| < 1\right\}\right)_{n\in\mathbb{N}}.$$

We study the generalization of NC ball maps to these pencil balls $\mathcal{B}_L$, and call them "pencil ball maps". We show that every $\mathcal{B}_L$ has a minimal dimensional (in a certain sense) defining pencil $\tilde{L}$. Up to normalization, a pencil ball map is the direct sum of $\tilde{L}$ with an NC analytic map of the pencil ball into the ball. That is, pencil ball maps are simple, in contrast to the classical result of D'Angelo (1993) [7, Chapter 5] showing there is a great variety of such analytic maps from $\mathbb{C}^g$ to $\mathbb{C}^m$ when $g \ll m$. To prove our main theorem, this paper uses the results of our previous paper (Helton et al. (2009) [9]) plus entirely different techniques, namely, those of completely contractive maps. What we do here is a small piece of the bigger puzzle of understanding how Linear Matrix Inequalities (LMIs) behave with respect to noncommutative change of variables.

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1. Introduction

Given positive integers n, d, d' and g, let $\mathbb{C}^{d'\times d}$ denote the $d'\times d$ matrices with complex coefficients and $(\mathbb{C}^{n\times n})^g$ the set of g-tuples of $n\times n$ matrices. For $A_1,\dots,A_g\in\mathbb{C}^{d'\times d}$, the expression

$$L(x)=\sum_{j=1}^{g} A_j x_j, \tag{1.1}$$

is a **homogeneous linear pencil**. (Often the term linear pencil refers to an *affine* linear function; i.e., a sum of a constant term plus a homogeneous linear pencil.) Given $X=\operatorname{col}(X_1,\dots,X_g)\in(\mathbb{C}^{n\times n})^g$, define

$$L(X)=\sum_{j=1}^{g} A_j\otimes X_j. \tag{1.2}$$

Let $\mathcal{B}_L(n)=\{X\in(\mathbb{C}^{n\times n})^g\colon\ \|L(X)\|<1\}$ and let $\mathcal{B}_L$ denote the sequence $(\mathcal{B}_L(n))_{n\in\mathbb{N}}$. Similarly, let $\mathcal{M}_{\ell',\ell}=((\mathbb{C}^{n\times n})^{\ell'\times\ell})_{n\in\mathbb{N}}$. The main result of this paper describes analytic mappings f from the **pencil ball** $\mathcal{B}_L$ to $\mathcal{M}_{\ell',\ell}$ that preserve the boundary in the sense described at the end of Section 1.1 below.

In the remainder of this introduction, we give the definitions and background necessary for a precise statement of the result, and provide a guide to the body of the paper.

1.1. Formal power series

Let $x=(x_1,\dots,x_g)$ be a g-tuple of noncommuting indeterminates and let $\langle x\rangle$ denote the set of all **words** in x. This includes the empty word denoted by 1. The length of a word $w\in\langle x\rangle$ will be denoted $|w|$. For an abelian group R we use $R\langle x\rangle$ to denote the abelian group of all (finite) sums of **monomials** (these are elements of the form rw for $r\in R$ and $w\in\langle x\rangle$).

Given positive integers ℓ, ℓ', a **formal power series** f in x with $\mathbb{C}^{\ell'\times\ell}$ coefficients is an expression of the form

$$f=\sum_{m=0}^{\infty}\sum_{\substack{w\in\langle x\rangle\\ |w|=m}} f_w w=\sum_{m=0}^{\infty} f^{(m)}, \tag{1.3}$$

where $f_w\in\mathbb{C}^{\ell'\times\ell}$ and $f^{(m)}\in\mathbb{C}^{\ell'\times\ell}\langle x\rangle$ is the **homogeneous component** of degree m of f, that is, the sum of all monomials in f of degree m. For $X\in(\mathbb{C}^{n\times n})^g$, $X=\operatorname{col}(X_1,\dots,X_g)$ and a word

$$w=x_{j_1}x_{j_2}\cdots x_{j_m}\in\langle x\rangle,$$

let

$$w(X)=X_{j_1}X_{j_2}\cdots X_{j_m}\in\mathbb{C}^{n\times n}.$$

Define

$$f(X)=\sum_{m=0}^{\infty}\sum_{\substack{w\in\langle x\rangle\\ |w|=m}} f_w\otimes w(X),$$

provided the series converges (summed in the indicated order). The function f is **analytic** on $\mathcal{B}_L$ if for each n and $X\in\mathcal{B}_L(n)$ the series $f(X)$ converges. Thus, in this case, the formal power series f determines a mapping from $\mathcal{B}_L(n)$ to $(\mathbb{C}^{n\times n})^{\ell'\times\ell}$ for each n which is expressed by writing $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$.

The analytic function $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ is **contraction-valued** if $\|f(X)\|\leqslant 1$ for each $X\in\mathcal{B}_L$; i.e., if the values of f are contractions. Let $\partial\mathcal{B}_L(n)$ denote the set of all $X\in(\mathbb{C}^{n\times n})^g$ with $\|L(X)\|=1$. If $f:\mathcal{B}_L\to\mathcal{M}_{\ell',\ell}$ is contraction-valued and $X\in\partial\mathcal{B}_L(n)$, then, by Fatou's theorem, the analytic function $f_X:\mathbb{D}\to(\mathbb{C}^{n\times n})^{\ell'\times\ell}$ defined by $f_X(z)=f(zX)$ has boundary values almost everywhere; i.e., $f(\exp(it)X)$ is defined for almost every t. (We use $\mathbb{D}$ to denote the unit disc $\{z\in\mathbb{C}\colon\ |z|<1\}$.) The contraction-valued function f is a **pencil ball map** if $\|f(\exp(it)X)\|=1$ a.e. for every $X\in\partial\mathcal{B}_L$. Here the boundary $\partial\mathcal{B}_L$ of the pencil ball $\mathcal{B}_L$ is the sequence $(\partial\mathcal{B}_L(n))_{n\in\mathbb{N}}$.

1.2. The main result

The homogeneous linear pencil L is **nondegenerate**, if it is one–one in the sense that

$$\forall X\in\left(\mathbb{C}^{n\times n}\right)^g\colon\quad \left(L(X)=0\ \Rightarrow\ X=0\right) \tag{1.4}$$

for all $n\in\mathbb{N}$.