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Some properties of a generalized Hamy symmetric function and its applications

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ABSTRACT

This paper is concerned with the generalized Hamy symmetric function

$$\sum\nolimits_n (x, r; f) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} f\left(\prod_{j=1}^{r} x_{i_j}^{\frac{1}{r}} \right),$$

where f is a positive function defined in a subinterval of $(0, +\infty)$. Some properties, including Schur-convexity, geometric Schur-convexity and harmonic Schur-convexity are investigated. As applications, several inequalities are obtained, some of which extend the known ones.

1. Introduction

Throughout this paper, let R_+ denote the set of all positive real numbers and R_+^n its n-product. For $\Omega \subseteq R_+^n$, let

$$\ln \Omega = \left\{ (\ln x_1, \ln x_2, \ldots, \ln x_n) \,\middle|\, x = (x_1, x_2, \ldots, x_n) \in \Omega \right\}$$

and

$$1/\Omega = \left\{ (1/x_1, 1/x_2, \ldots, 1/x_n) \,\middle|\, x = (x_1, x_2, \ldots, x_n) \in \Omega \right\}.$$

For a positive n-tuple $x = (x_1, x_2, \ldots, x_n) \in R_+^n$, Hamy [12] introduced the symmetric function

$$F_n(x, r) = F_n(x_1, \ldots, x_n; r) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} \left(\prod_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}}, \quad r = 1, 2, \ldots, n. \tag{1.1}$$

In Hamy's honor, the above function is called Hamy symmetric function. Corresponding to this function is the r-th order Hamy mean

$$\sigma_n(x, r) = \frac{1}{\binom{n}{r}} \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} \left(\prod_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}}, \quad r = 1, 2, \ldots, n, \tag{1.2}$$

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where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. It is obvious that $\sigma_n(x,1)$ is the arithmetic mean

$$A_n(x) = A_n(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n},$$

and $\sigma_n(x,n)$ is the geometric mean

$$G_n(x) = G_n(x_1, x_2, \dots, x_n) = \sqrt[n]{x_1 x_2 \dots x_n}.$$

There are some papers on Hamy symmetric function and its mean. For example, Hara et al. [13] established the following refinement of the classical arithmetic and geometric means inequality:

$$G_n(x) = \sigma_n(x,n) \leqslant \sigma_n(x,n-1) \leqslant \cdots \leqslant \sigma_n(x,2) \leqslant \sigma_n(x,1) = A_n(x). \tag{1.3}$$

The paper [17] by Ku et al. contains some interesting inequalities including the fact that $(\sigma_n(x,r))^r$ is log-concave. For more details, please refer to the book [6] by Bullen. In 2006, Guan [8] investigated Schur-convexity of Hamy symmetric function $F_n(x,r)$ and some inequalities were also obtained by use of the theory of majorization.

Recently, Guan [9] defined a generalized Hamy symmetric function of the form

$$\sum\nolimits_n (x, r; f) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n} f\left(\prod_{j=1}^{r} x_{i_j}^{\frac{1}{r}}\right), \quad r = 1, 2, \dots, n, \tag{1.4}$$

where f is a positive function defined in a subinterval of $(0, +\infty)$. The author investigated the geometric Schur-convexity of $\sum_n(x, r; f)$ when f is a multiplicatively convex function, i.e., *GG*-convex function.

The main purpose of this paper is to investigate further Schur-convexity, geometric Schur-convexity, and harmonic Schur-convexity of $\sum_n(x, r; f)$. As applications, some inequalities are established by use of the theory of majorization. Our results improve the known ones.

The notation of Schur-convex function was introduced by I. Schur in 1923 [24]. It has many important applications in analytic inequalities [4,8,10,14,19,25], combinatorial optimization [15], isoperimetric problem for polytopes [27], gamma and digamma functions [20], and other related fields. For a historical development of this kind of functions and the fruitful applications to statistics, economics and other applied fields, refer to the popular book by Marshall and Olkin [19].

Definition 1.1. (See [10,19,24–26].) A real-valued function ϕ defined on a set $\Omega \subseteq R^n$ $(n \geqslant 2)$ is said to be a Schur-convex function on Ω if

$$\phi(x_1, x_2, \dots, x_n) \leqslant \phi(y_1, y_2, \dots, y_n)$$

for each pair of n-tuples $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ on Ω, such that x is majorized by y (in symbols $x \prec y$), that is,

$$\sum_{i=1}^{m} x_{[i]} \leqslant \sum_{i=1}^{m} y_{[i]}, \quad m = 1, 2, \dots, n-1, \quad \text{and} \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the ith largest component in x. ϕ is called Schur-concave if $-\phi$ is Schur-convex.

The notation of geometric convexity was introduced by Montel [22] and investigated by Anderson et al. [3], Guan [11] and Niculescu [23]. The geometric Schur-convexity was investigated by Chu et al. [7], Guan [9], and Niculescu [23]. We also note that the authors use the term "Schur-multiplicative (geometric) convexity". However, we here point out that the term "geometric Schur-convexity" is more appropriate. As a matter of fact, from [7,9,11,22,23], we have no difficulty to find that f being geometric convex in convexity theory means that the function $x \mapsto \log f(e^x)$ is convex and that "Schur-multiplicative (geometric) convexity" of ϕ is equivalent to Schur-convexity of the function $x \mapsto \phi(e^x)$, which in turn, for positive functions, is equivalent to Schur-convexity of the function $x \mapsto \log\phi(e^x)$. Thus, we give an alternative definition of geometric Schur-convexity.

Definition 1.2. Let $\Omega \subseteq R_+^n$ $(n \geqslant 2)$ be a set. A real-valued function $\phi : \Omega \to R$ is called a geometrically Schur-convex function on Ω if the function $x \mapsto \phi(e^x)$ is Schur-convex on $\ln\Omega$. ϕ is called geometrically Schur-concave if $-\phi$ is geometrically Schur-convex.

Recently, Xia et al. [26] introduced the notion of harmonically Schur-convex function and some interesting inequalities were obtained.

Definition 1.3. (See [26].) Let $\Omega \subseteq R_+^n$ $(n \geqslant 2)$ be a set. A real-valued function ϕ defined on Ω is called a harmonically Schur-convex function if the function $x \mapsto \phi(1/x)$ is Schur-convex on $1/\Omega$. ϕ is called a harmonically Schur-concave function on Ω if $-\phi$ is harmonically Schur-convex.