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Extension of the operator of best polynomial approximation in $L^p(\Omega)$ ☆

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ABSTRACT

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. In this paper we extend the operator of the best generalized polynomial approximation from the space $L^p(\Omega)$ to the space $L^{p-1}(\Omega)$, $1 < p < \infty$, as the unique operator preserving the property of continuity. The case $p = 1$ is also considered.

1. Introduction and notation

Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. As usual, $L^p = L^p(\Omega)$, $1 \leqslant p < \infty$, will denote the system of all equivalence classes of measurable functions $f : \Omega \to \mathbb{R}$ with $\|f\|_p := (\int_\Omega |f|^p \, d\mu)^{1/p} < \infty$, and $\|f\|_\infty = \sup\operatorname{ess}\{|f(x)|: x \in \Omega\}$. Let $S \subset L^p$ be the space generated by $\{\psi_j\}_{j=1}^s$, i.e., $S = \{\sum_{j=1}^s a_j \psi_j: a_j \in \mathbb{R}\}$, $s \in \mathbb{N}$. We assume that the functions ψ_j are linearly independent, measurable and bounded on Ω. Sometimes in literature regarding approximation theory, S is termed as a space of generalized polynomials (see [2, p. 72]). In the particular case that $\psi_j(x) = x^{j-1}$, $x \in \mathbb{R}$, $1 \leqslant j \leqslant s$, S is the space of algebraic polynomials of degree $s - 1$.

A polynomial (generalized) $P \in S$ is called a best approximant to f if $\|f - P\|_p \leqslant \|f - Q\|_p$ for every $Q \in S$. If $p > 1$, it is well known (see [12, p. 56]) that, $P \in S$ is a best approximant to f from S if and only if

$$\int_\Omega |f - P|^{p-1} \operatorname{sgn}(f - P) Q \, d\mu = 0, \quad \text{for every } Q \in S. \tag{1.1}$$

Such a polynomial $P =: T(f)$ there always exists and it is unique. It is called the best $\|.\|_p$-approximant to f from S. We observe that the left member of (1.1) is defined even if $f \in L^{p-1}$.

The main goal of this paper is to prove that, for each $f \in L^{p-1}$, $p > 1$, there exists a polynomial $P \in S$ satisfying (1.1) (see Section 3). This polynomial will be called an *extended polynomial approximant*. In the case $p = 2$, we can find the explicit expression of P for every $f \in L^1$. In fact, here (1.1) is a system of linear equations where the polynomial coefficients are the unknowns, with unique solution the set of coefficients of the extended polynomial approximant.

We will prove in Section 4 the uniqueness of the extended polynomial approximant, which will allow us to extend the operator of best polynomial approximation, T, to an operator $\overline{T} : L^{p-1} \to S$. We will show that $\overline{T}$ is continuous and, in consequence, it is the unique extension of T preserving the property of continuity.

In Section 5 we will also consider the case $p = 1$, and under certain condition on the subspace S, we will extend the operator of best polynomial approximation to an adequate space L_0, which contains L^q for every $q > 0$. In the particular

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case $s = 1$, and $\psi_1 = 1$, $S_0 := S$ is the space of constant functions and we can take a space greater than L_0, that is, the space of finite a.e. measurable functions. We will study continuity properties of the operator $\overline{T}$, and we will prove that $\overline{T}(f)$ is a compact and convex set.

In the case of the subspace S_0, the operator $\overline{T}$ was studied in [7] for $1 \leqslant p < \infty$, and several properties of $\overline{T}$ were given. In that paper the authors extended the best constant approximation operator to L^{p-1} if $p > 1$, and to the space of finite a.e. measurable functions if $p = 1$. Later, other authors in [3] and [4] considered the operator $\overline{T}$ defined in Orlicz spaces, and more recently in [6] the operator $\overline{T}$ was studied in Orlicz–Lorentz spaces. In this literature, inequalities for the maximal function induced by the extended constant approximation operator were studied.

For $S = S_0$ and $p > 1$, both the existence and the uniqueness of P satisfying (1.1) are easy to prove, as we can see in [7].

In relation to the extension of the operator of best isotonic approximation can be seen in [1] for Orlicz spaces, and in [8] for the L^1 space. In [9] the authors consider a finite measure space and, using the monotonicity of the operator, they extend the operator of best isotonic approximation from L^p to L^{p-1}, $p > 1$. First the operator is extended to functions $f \in L^{p-1}$, which are bounded from below, as the limit of the best approximants of the functions $f \wedge n$, as $n \to \infty$, and later as the limit of the best approximants to $g \vee -n$, as $n \to \infty$, for arbitrary $g \in L^{p-1}$, where $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$, $a, b \in \mathbb{R}$. The existence of such limits is a consequence of the monotonicity of the operator.

In this paper we will utilize the last technique to extend the operator of best polynomial approximation, though this method will be more complex than utilizing isotonic approximation, because generally we have not the monotonicity of the operator. For example, when S is the space of algebraic polynomials of degree at most $s - 1$, the best approximation operator is not monotone if $s \geqslant 2$.

2. Preliminary results

We write $\phi(x) = |x|^{p-1}\operatorname{sgn}(x)$, for $x \in \mathbb{R}$, $p \geqslant 1$, and we note that ϕ is a continuous and strictly increasing function when $p > 1$.

Throughout Sections 2 to 4, we consider $p > 1$ fixed. Let $f \in L^{p-1}$ be bounded from below. As

$$\phi\big((f - P)(x)\big) = \phi\big(\big((f - a) - (P - a)\big)(x)\big), \quad \text{for every } a \in \mathbb{R},$$

for our purposes we can assume, without loss of generality, that f is bounded from below by 0.

For $n \in \mathbb{N}$, we consider the following function in L^p, $f_n = f \wedge n$. Let $P_n \in S$ be the best $\|.\|_p$-approximant to f_n from S. We will prove that the sequence (P_n) has a subsequence uniformly bounded on Ω.

We begin with an auxiliary lemma.

Lemma 2.1. *There exists $n_0 \in \mathbb{N}$ such that $\|P_n\|_\infty < n$ for every $n \geqslant n_0$.*

Proof.

$$\left\| \frac{f_n}{n} \right\|_p^p = \int_{\{f \geqslant n\}} 1\, d\mu + \int_{\{f<n\}} \frac{|f|^p}{n^p}\, d\mu. \tag{2.1}$$

By the Chebyshev's inequality, $\mu(\{f \geqslant n\}) \to 0$, as $n \to \infty$. On the other hand,

$$\int_{\{f<n\}} |f|^p\, d\mu = \int_{\{f<n\}} |f|^{p-1}|f|\, d\mu \leqslant n\|f\|_{p-1}^{p-1}.$$

In consequence, we get

$$\int_{\{f<n\}} \frac{|f|^p}{n^p}\, d\mu \leqslant \frac{1}{n^{p-1}}\|f\|_{p-1}^{p-1}. \tag{2.2}$$

Since we have assumed that $p > 1$, from (2.1) and (2.2) we have

$$\left\| \frac{f_n}{n} \right\|_p \to 0, \quad \text{as } n \to \infty. \tag{2.3}$$

If the lemma is not true, then there exists a subsequence $(n_k) \subset \mathbb{N}$, such that $\|P_{n_k}\|_\infty \geqslant n_k$. The equivalence of the norms in S implies that there exists $M > 0$ such that $\|P_{n_k}\|_\infty \leqslant M\|P_{n_k}\|_p$, for every $k \in \mathbb{N}$. Considering also that the best $\|.\|_p$-approximant to f_{n_k} belongs to the ball in L^p centered at zero and of radius $2\|f_{n_k}\|_p$, the following inequality holds

$$n_k \leqslant \|P_{n_k}\|_\infty \leqslant M\|P_{n_k}\|_p \leqslant 2M\|f_{n_k}\|_p. \tag{2.4}$$

So, (2.4) contradicts (2.3). $\square$