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Gaps of operators via rank-one perturbations

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ABSTRACT

In this note we consider rank-one perturbations of weighted shifts to examine distinctions between various sorts of weak hyponormalities, including p-hyponormality, p-paranormality, and absolute-p-paranormality. Our examples enable us to add to the small collection of examples that exhibit the gaps between these classes.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite-dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. The study of partial normalities such as p-hyponormality and other weak hyponormalities has been considered for some 30 years (see [10,16]). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* $(0 < p < \infty)$ if $(T^*T)^p \geqslant (TT^*)^p$. And $T \in \mathcal{L}(\mathcal{H})$ is ∞-hyponormal if T is p-hyponormal for all $p \in (0, \infty)$. In particular, if $p = \frac{1}{2}$, then T is said to be *semi-hyponormal* [16]. Recall that an operator $T \in \mathcal{L}(\mathcal{H})$ has the unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry satisfying $\ker U = \ker |T| = \ker T$ and $\ker U^* = \ker T^*$. For each $p > 0$, an operator T is *p-paranormal* if $\||T|^p U|T|^p x\| \geqslant \||T|^p x\|^2$ for all unit vectors x in $\mathcal{H}$. Every q-paranormal operator is p-paranormal for $q \leqslant p$. And T is *absolute-p-paranormal* if $\||T|^p Tx\| \geqslant \|Tx\|^{p+1}$ for all unit vectors x in $\mathcal{H}$. Observe that absolute-1-paranormality and 1-paranormality coincide; we call this property simply *paranormality*. Note that every absolute-q-paranormal operator is absolute-p-paranormal for $q \leqslant p$ [10]. The implications among classes of operators mentioned above are as follows:

- p-hyponormal $\Rightarrow$ p-paranormal $\Rightarrow$ absolute-p-paranormal $(0 < p < 1)$;
- p-hyponormal $\Rightarrow$ absolute-p-paranormal $\Rightarrow$ p-paranormal $(p > 1)$.

Since examples for these operator classes are not abundant, it is worthwhile to develop examples to distinguish these classes. In [6,13,14], some block matrix operators were considered to exemplify the above classes, but it was proved in their models that p-paranormality is equivalent to absolute-p-paranormality. Also, models of composition operators were discussed in [3,14,2,4], and [7], to exemplify these various partial normality classes, but it also was shown that for these operators the two partial normalities coincide [2]. There do not exist examples showing that p-paranormality and absolute-q-paranormality are distinct except an example in [10, p. 179] that there is an absolute-2-paranormal operator which is not

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1-paranormal. But examples showing that the two notions of p-paranormality and absolute-p-paranormality are distinct for all p are not yet known. So it is worthwhile to seek such examples to show these two notions are distinct.

For non-zero vectors x and y in $\mathcal{H}$, we consider the rank-one operator $x \otimes y$ defined by $(x \otimes y)(z) = \langle z, y\rangle x$ for all z in $\mathcal{H}$. Let W_α be a weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$. Then $W_\alpha + t(x \otimes y)$ is a rank-one perturbation of W_α with the parameter $t \in (0, \infty)$. In this note we discuss some special rank-one perturbations of weighted shifts which can serve to distinguish these classes.

The paper consists of three sections. In Section 2 we characterize p-hyponormality for rank-one perturbations of a weighted shift, and obtain examples showing the classes of p-hyponormal operators are distinct in $p > 0$. In Section 3, we also characterize absolute-p-paranormality and p-paranormality for the rank-one perturbations of weighted shifts considered, which provide examples showing these two classes are distinct.

Some of the calculations in this paper were obtained through computer experiments using the software tool *Mathematica* [15].

2. *p*-Hyponormalities

Let W_α be a weighted shift with weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ of nonnegative real numbers. Let $\{e_i\}_{i=0}^{\infty}$ be an orthonormal basis for $\mathcal{H} = \ell^2(\mathbb{Z}_+)$. Obviously, W_α is hyponormal if and only if W_α is p-hyponormal for any [some] $p \in (0, \infty)$. In particular, W_α is normal if and only if $\alpha_n = 0$ for all $n \geqslant 0$, and W_α is quasinormal if and only if $\alpha_n(\alpha_{n+1}^2 - \alpha_n^2) = 0$ for all $n \geqslant 0$. Hence weighted shifts cannot separate classes of p-hyponormal operators. But rank-one perturbations of weighted shifts with a positive real parameter separate the classes of p-hyponormal operators completely.

For a fixed $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we consider a rank-one perturbation of a weighted shift

$$T(k,t) := W_\alpha + t(e_k \otimes e_k) \tag{2.1}$$

with parameter $t \in (0, \infty)$.

Throughout this paper we extend weight sequence $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ with $\alpha_{-1} = \alpha_{-2} = \alpha_{-3} = 0$ if such notations are convenient. For example, we set $\alpha_{-1} = \alpha_{-2} = 0$ in Theorem 2.1 below.

Theorem 2.1. *Let $T(k,t)$ be as in* (2.1)*. Suppose that $p \in (0, \infty)$. Then the following assertions hold*:

(i) *if $\alpha_{k-1} > 0$ and $\alpha_k > 0$ (which happens only if $k \geqslant 1$), then $T(k,t)$ is p-hyponormal if and only if $\alpha_i \leqslant \alpha_{i+1}$ $(0 \leqslant i \leqslant k-3$; $i \geqslant k+1)$ and it holds that*

$$\delta_{11} > 0, \qquad \delta_{11}\delta_{22} - \delta_{12}^2 > 0, \quad \text{and} \quad \delta_{33}\big(\delta_{11}\delta_{22} - \delta_{12}^2\big) - \delta_{11}\delta_{23}^2 \geqslant 0, \tag{2.2}$$

where

$$\begin{aligned}
&\delta_{11} = -\alpha_{k-2}^{2p} + \big\{\big(t^2 + \alpha_k^2 - \alpha_{k-1}^2 + \gamma_k\big)\lambda_k^p + \big(\alpha_{k-1}^2 - t^2 - \alpha_k^2 + \gamma_k\big)\mu_k^p\big\}/(2\gamma_k);\\
&\delta_{12} = \delta_{21} = t\alpha_{k-1}\big(\mu_k^p - \lambda_k^p\big)/\gamma_k; \qquad \delta_{22} = \big(\alpha_k^2 - \alpha_{k-1}^2\big)\big(\mu_k^p - \lambda_k^p\big)/\gamma_k;\\
&\delta_{23} = \delta_{32} = t\alpha_k\big(\lambda_k^p - \mu_k^p\big)/\gamma_k;\\
&\delta_{33} = \alpha_{k+1}^{2p} - \big\{\big(t^2 + \alpha_{k-1}^2 - \alpha_k^2 + \gamma_k\big)\lambda_k^p + \big(\alpha_k^2 - t^2 - \alpha_{k-1}^2 + \gamma_k\big)\mu_k^p\big\}/(2\gamma_k);\\
&\lambda_k = \big(t^2 + \alpha_{k-1}^2 + \alpha_k^2 - \gamma_k\big)/2; \qquad \mu_k = \big(t^2 + \alpha_{k-1}^2 + \alpha_k^2 + \gamma_k\big)/2;\\
&\gamma_k = \big[\big(t^2 + \alpha_{k-1}^2 + \alpha_k^2\big)^2 - 4(\alpha_{k-1}\alpha_k)^2\big]^{1/2},
\end{aligned} \tag{2.3}$$

(ii) *if $\alpha_{k-1} = 0$ and $\alpha_k > 0$, then $T(k,t)$ is p-hyponormal if and only if $\alpha_i = 0$ $(0 \leqslant i \leqslant k-2)$, $\alpha_{i+k+1} \geqslant \alpha_{i+k}$ $(i \in \mathbb{N})$, and $\alpha_{k+1}^2 \geqslant \alpha_k^2 + t^2$,*

(iii) *if $\alpha_{k-1} = 0$ and $\alpha_k = 0$, then $T(k,t)$ is p-hyponormal if and only if $\alpha_i = 0$ $(0 \leqslant i \leqslant k-2)$ and $\alpha_{i+1} \geqslant \alpha_i$ $(i \geqslant k+1)$.*

Proof. (i) By simple computations, we have that

$$T(k,t)^*T(k,t) = \operatorname{Diag}\big\{\alpha_0^2, \ldots, \alpha_{k-2}^2, A_k, \alpha_{k+2}^2, \ldots\big\}$$

and

$$T(k,t)T(k,t)^* = \operatorname{Diag}\big\{0, \alpha_0^2, \ldots, \alpha_{k-3}^2, B_k, \alpha_{k+1}^2, \ldots\big\}$$

with

$$A_k = \begin{bmatrix} \alpha_{k-1}^2 & t\alpha_{k-1} & 0 \\ t\alpha_{k-1} & \boxed{t^2 + \alpha_k^2} & 0 \\ 0 & 0 & \alpha_{k+1}^2 \end{bmatrix} \quad \text{and} \quad B_k = \begin{bmatrix} \alpha_{k-2}^2 & 0 & 0 \\ 0 & \boxed{t^2 + \alpha_{k-1}^2} & t\alpha_k \\ 0 & t\alpha_k & \alpha_k^2 \end{bmatrix}, \tag{2.4}$$