J. Math. Anal. Appl. 376 (2011) 602–612

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The e-support function of an e-convex set and conjugacy for e-convex functions

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ARTICLE INFO

Article history:
Received 8 March 2010
Available online 5 November 2010
Submitted by Richard M. Aron

Keywords:
Evenly convex set
Evenly convex function
Support function
Convex conjugation

ABSTRACT

A subset of a locally convex space is called e-convex if it is the intersection of a family of open halfspaces. An extended real-valued function on such a space is called e-convex if its epigraph is e-convex. In this paper we introduce a suitable support function for e-convex sets as well as a conjugation scheme for e-convex functions.

1. Introduction

A set C in a locally convex real topological vector space X is said to be *evenly convex* (or, in brief, *e-convex*), if it is the intersection of a family of open halfspaces. These sets were introduced long time ago by Fenchel [1], and a detailed study of their fundamental properties was recently provided in [2,4,5]. A function $f : X \to \overline{\mathbb{R}}$, with $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, is said to be *evenly convex* (or, briefly, *e-convex*) if its epigraph $\operatorname{epi} f := \{(x,\lambda) \in X \times \mathbb{R}\colon\ f(x) \leqslant \lambda\}$ is e-convex. The class of e-convex functions has been introduced in the recent work [8].

The aim of this paper is twofold: to provide a suitable support function for e-convex sets and a conjugation scheme for e-convex functions. The classical support function of Convex Analysis is not appropriate for e-convex sets, since different e-convex sets may have the same closure and therefore identical support functions. In Section 3 we will introduce the so-called e-support function of a set $C \subset X$, in such a way that when C is e-convex its e-support function contains whole information on the set. Similarly, the classical Fenchel conjugation theory is not suitable for e-convex functions, since different e-convex functions may have the same lower semicontinuous hull and hence identical second conjugates. In Section 4 we will provide a conjugation scheme for extended real-valued functions on X, which in the case of an e-convex function yields the second conjugate identical to the original function. This conjugation scheme will be based on some new characterizations of e-convex functions as suprema of suitably introduced elementary functions, which we will present in Section 2.

Throughout this paper we will adopt the standard terminology and notation of convex analysis. In particular, we will say that f is *proper* if it is not identically $+\infty$ and does not take the value $-\infty$; otherwise it is called *improper*. The function f is said to be *lsc* (lower semicontinuous) at a point if it coincides with its lower semicontinuous hull at that point. If f

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1 This author has been supported by MICINN of Spain, Grant MTM2008-06695-C03-03, the Barcelona Graduate School of Economics and the Government of Catalonia. He is affiliated to MOVE (Markets, Organizations and Votes in Economics).
2 This author has been supported by FPI Program of MICINN of Spain, Grant BES-2006-14041.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.10.058

is lsc at every point of a set then f is said to be lsc on that set. One says that f is *sublinear* if it is convex and positively homogeneous. We will denote by X^* the topological dual space of X; as X is assumed to be locally convex one has $X^* \neq \{0\}$ and, consequently, there are open halfspaces in X. Moreover, by Hahn–Banach Theorem all convex sets which are open or closed are e-convex. We will denote by $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}$ the duality product: $\langle x, x^*\rangle = x^*(x)$ for $(x, x^*) \in X \times X^*$. The indicator function of $C \subset X$ is $\delta_C : X \to \overline{\mathbb{R}}$, defined by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$. The support function of C is $\sigma_C : X^* \to \overline{\mathbb{R}}$, defined by $\sigma_C(x^*) = \sup\{\langle x, x^*\rangle\colon x \in C\}$. The closure and the relative interior of C will be denoted by $\operatorname{cl} C$ and $\operatorname{ri} C$, respectively. The domain of $f : X \to \overline{\mathbb{R}}$ is the set $\operatorname{dom} f := \{x \in X\colon f(x) < +\infty\}$. The convex conjugate and the Fenchel subdifferential of f will be denoted by f^* and ∂f, respectively, and $\operatorname{co} C$ will stand for the convex hull of C. One says that f is subdifferentiable at $x_0 \in X$ if $\partial f(x_0) \neq \emptyset$. The e-convex hull $\operatorname{eco} C$ of $C \subset X$ is the smallest e-convex set that contains C; its existence follows from the fact that X is e-convex and the class of e-convex sets is closed under intersection. Since every closed convex set is e-convex, one clearly has $\operatorname{co} C \subset \operatorname{eco} C \subset \operatorname{cl}\operatorname{co} C$. Since these three sets generate the same affine manifold, assuming that C is convex and taking relative interiors in the latter inclusions we get $\operatorname{ri} C \subset \operatorname{ri}\operatorname{eco} C \subset \operatorname{ri}\operatorname{cl} C = \operatorname{ri} C$ [9, Theorem 1.1.2]; hence $\operatorname{ri} C = \operatorname{ri}\operatorname{eco} C$ if C is convex. An easy consequence of this equality is that if a convex set $C \subset \mathbb{R}^n$ is not relatively open, then its e-convex hull is not relatively open either. The e-convex hull of f is defined as the largest e-convex minorant of f:

$$\operatorname{eco} f := \sup\{g\colon\ g \text{ is e-convex and } g \leqslant f\};$$

this function is indeed e-convex, since the class of e-convex functions is closed under pointwise supremum.

2. Some new characterizations of e-convex functions

The following characterization theorems for e-convex functions are proved in [8]:

Theorem 1. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be such that $f(x_0) = -\infty$ for some $x_0 \in \mathbb{R}^n$. Then*

$$f \text{ is e-convex} \quad \Leftrightarrow \quad \operatorname{dom} f \text{ is e-convex and } f \text{ is identically } -\infty \text{ on } \operatorname{dom} f.$$

The only improper function not covered by the preceding theorem is the constant function $+\infty$, which is obviously e-convex.

Theorem 2. *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a proper convex function. Then*

$$f \text{ is e-convex} \quad \Leftrightarrow \quad f \text{ is lsc on } \operatorname{eco}(\operatorname{dom} f).$$

In this section we will give some new characterizations of e-convex functions defined on locally convex spaces. For $f : X \to \overline{\mathbb{R}}$, we will use the simplified notation

$$M_f := \operatorname{eco}(\operatorname{dom} f).$$

We will need the following definition:

Definition 3. Let $f : X \to \overline{\mathbb{R}}$ be an arbitrary function and $C \subset X$. We will say that $a : X \to \overline{\mathbb{R}}$ is C-affine if there exist $c \in X^*$ and $\alpha \in \mathbb{R}$ such that

$$a(x) = \begin{cases} \langle x, c\rangle - \alpha & \text{if } x \in C,\\ +\infty & \text{if } x \notin C. \end{cases}$$

We will make an extensive use of the set of all M_f-affine minorants of f:

$$\mathcal{H}_f := \{a : X \to \overline{\mathbb{R}}\colon\ a \text{ is } M_f\text{-affine and } a \leqslant f\}.$$

Remark 4. Notice that if $f \equiv +\infty$ then $\operatorname{dom} f = \emptyset$ and hence the only M_f-affine function is f. On the other hand, if f is improper but $f \not\equiv +\infty$ then $\mathcal{H}_f = \emptyset$.

Proposition 5. *If $C \subset X$ is e-convex, then every C-affine function is e-convex.*

Proof. If $a : X \to \overline{\mathbb{R}}$ is C-affine, its epigraph is the intersection of a closed halfspace in $X \times \mathbb{R}$ with $C \times \mathbb{R}$, which is clearly an e-convex set. $\square$

From the preceding proposition it follows that, for any function $f : X \to \overline{\mathbb{R}}$, every M_f-affine function is e-convex.