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Constructive factorization of some almost periodic triangular matrix functions with a quadrinomial off diagonal entry ☆

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ABSTRACT

Using an abbreviation e_μ to denote the function $e^{i\mu x}$ on the real line $\mathbb{R}$, let $G = \left[\begin{smallmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{smallmatrix}\right]$, where f is a linear combination of the functions e_α, e_β, $e_{\alpha-\lambda}$, $e_{\beta-\lambda}$ with some $(0 <)\ \alpha, \beta < \lambda$. The criterion for G to admit a canonical factorization was established recently by Avdonin, Bulanova and Moran (2007) [1]. We give an alternative approach to the matter, proving the existence (when it does take place) via deriving explicit factorization formulas. The non-existence of the canonical factorization in the remaining cases then follows from the continuity property of the geometric mean.

1. Introduction

We denote by *APP* the algebra of all *almost periodic polynomials*, that is, finite linear combinations of functions $e_\lambda := e^{i\lambda x}$, where $\lambda \in \mathbb{R}$. The *AP* algebra is the closure of *APP* with respect to the uniform norm while the *APW* algebra is the closure of *APP* with respect to a stronger norm,

$$\left\| \sum_j c_j e_{\lambda_j} \right\|_W = \sum_j |c_j|, \quad c_j \in \mathbb{C}.$$

For any $f \in AP$ there exists the Bohr *mean value*

$$\mathbf{M}(f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(x)\, dx.$$

The functions $f \in AP$ are defined uniquely by their *Bohr–Fourier series*

$$\sum_{\lambda \in \Omega(f)} \hat{f}(\lambda) e_\lambda \tag{1.1}$$

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where $\hat{f}(\lambda) = \mathbf{M}(fe_{-\lambda})$ are its *Bohr–Fourier coefficients* and

$$\Omega(f) := \{\lambda \in \mathbb{R}:\ \hat{f}(\lambda) \neq 0\}$$

is the *Bohr–Fourier spectrum* of f. For $f \in APW$ the series (1.1) converges to f uniformly and absolutely on $\mathbb{R}$.

Denote by $AP_\pm$ $(APW_\pm, APP_\pm)$ the subalgebra of AP (APW, APP) consisting of all functions f with $\Omega(f) \subset \mathbb{R}_\pm \cup \{0\}$, respectively.

If X is an algebra of scalar valued functions we denote by $X^{n\times n}$ the algebra of $n \times n$ matrices with entries in X. The Bohr mean, the Bohr–Fourier coefficients and spectrum for $f \in AP^{n\times n}$ are defined by the same formulas as in the scalar case.

Definition 1.1. A left AP factorization of an $n \times n$ matrix function G is the representation

$$G = G_+ D G_-^{-1}, \tag{1.2}$$

where $G_+^{\pm 1} \in AP_+^{n\times n}$, $G_-^{\pm 1} \in AP_-^{n\times n}$ and $D = \operatorname{diag}(e_{\lambda_1}, \dots, e_{\lambda_n})$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, n$.

The representation (1.2) is a left APW (APP) factorization of G if $G_+^{\pm 1} \in APW_+^{n\times n}$ $(APP_+^{n\times n})$, $G_-^{\pm 1} \in APW_-^{n\times n}$ $(APP_-^{n\times n})$. Factorization (1.2) is *canonical* if $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$, that is,

$$G = G_+ G_-^{-1}. \tag{1.3}$$

Factorization (1.3), when it exists, is defined up to the transformation $G_\pm \mapsto G_\pm Z$, where $Z \in \mathbb{C}^{n\times n}$ is invertible. Therefore, the quantity

$$\mathbf{d}(G) := \mathbf{M}(G_+)\mathbf{M}(G_-)^{-1}$$

is defined uniquely, whenever G admits a canonical AP factorization. It is called the *geometric mean* of G.

A detailed discussion of AP factorization and its applications to singular integral, Wiener–Hopf and convolution type equations can be found in [2]. We mention here only several facts, directly related to the subject at hand.

It easily follows from (1.3) that

$$\Omega(G_+^{\pm 1}) \subset [0, \sup \Omega(G^{\pm 1})], \qquad \Omega(G_-^{\pm 1}) \subset [\inf \Omega(G^{\mp 1}), 0], \tag{1.4}$$

whenever G admits a canonical left AP factorization. Moreover, if Σ is the smallest additive subgroup of $\mathbb{R}$ containing the Bohr–Fourier spectrum $\Omega(G)$ of G, then also (see [3,4])

$$\Omega(G_+^{\pm 1}) \subset \Sigma, \qquad \Omega(G_-^{\pm 1}) \subset \Sigma. \tag{1.5}$$

A much deeper result is that if $G \in APW^{n\times n}$ admits a canonical AP factorization (1.3), then automatically $G_\pm \in APW^{n\times n}$ as well, that is, (1.3) is in fact an APW factorization. Moreover, the set of matrix functions $G \in AP^{n\times n}$ admitting a canonical AP factorization is open, and the function $G \mapsto \mathbf{d}(G)$ is continuous on it.

A key problem in AP factorization theory is a constructive existence criterion. Such a criterion is unknown even for matrix functions of the form

$$G = \begin{bmatrix} e_\lambda & 0 \\ f & e_{-\lambda} \end{bmatrix} \tag{1.6}$$

with $f \in AP$. Note that such matrices arise naturally when considering convolution type equations on finite intervals, λ (> 0) being the length of the interval and f describing the asymptotic of the Fourier transform of the kernel. Several results on the factorization of matrix functions (1.6) can be found in [2, Chapters 14 and 15], including some types of a trinomial f. In [1], an invertibility criterion was obtained for some difference operator on $L_2[0,1]$ which can be interpreted as a necessary and sufficient condition for (1.6) to admit a canonical AP factorization in the case when f is a quadrinomial given by

$$f = C_1 e_\alpha + C_{-1} e_{\alpha-1} + C_2 e_\beta + C_{-2} e_{\beta-1}$$

with $0 < \alpha < \beta < 1$ $(= \lambda)$. Such an interpretation was given in [5], where this result was also combined with the so-called Portuguese transformation to obtain factorability criteria for some other matrix functions of type (1.6). Observe, however, that results of [1], and therefore [5], do not provide constructive factorization formulas. The goal of our paper is to fill this gap. Thus, an explicit factorization is given for matrix function of the form (1.6) arising in construction of sampling and interpolating sequences for multi-band signals.

The structure of the factorization multiples, that is, their Bohr–Fourier spectra and the recursive formulas for the Bohr–Fourier coefficients, are derived in Section 2. The convergence question is settled in Section 3. It contains two main theorems, which cover all the cases when a canonical factorization exists. Finally, a short Section 4 consists of several remarks, including the proof of non-existence of the canonical factorization when conditions of Section 3 are not met.