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Real-parameter square-integrable solutions and the spectrum of differential operators ☆

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ABSTRACT

We continue to investigate the connection between the spectrum of self-adjoint ordinary differential operators with arbitrary deficiency index d and the number of linearly independent square-integrable solutions for real values of the spectral parameter λ. We show that if, for all λ in an open interval I, there are d linearly independent square-integrable solutions, then there is no continuous spectrum in I. This for any self-adjoint realization with boundary conditions which may be separated, coupled, or mixed. The proof is based on a new characterization of self-adjoint domains and on limit-point (LP) and limit-circle (LC) solutions established in an earlier paper.

1. Introduction

The spectrum of a self-adjoint ordinary differential operator in Hilbert space $H = L^2(J, w)$, $J = (a, b)$, is real, consists of eigenvalues of finite multiplicity and of essential spectrum. A number λ is an eigenvalue if the corresponding differential equation has a nontrivial solution which satisfies the boundary conditions. This happens 'coincidentally'. On the other hand, the essential spectrum is independent of the boundary conditions and thus depends only on the coefficients, including the weight function w, of the equation. This dependence is implicit and highly complicated. The coefficients and the weight function also determine the deficiency index d of the minimal operator $S_{\min}$ determined by the equation. For real-valued coefficients this is the number of linearly independent solutions in H for nonreal values of the spectral parameter λ and this number is independent of λ provided $\mathrm{Im}(\lambda) \neq 0$. For real values of λ the number of linearly independent solutions $r(\lambda)$ which lie in H is always less than or equal to d [27, Theorem 2]. In this paper we continue to explore the relationships between $r(\lambda)$ and the nondiscrete spectrum.

One such relationship is the well-known result [29] that if $r(\lambda) < d$, then λ is in the essential spectrum of every self-adjoint extension of $S_{\min}$. What if $r(\lambda) = d$? Questions of this kind date back to Hartman and Wintner [16] for the second order Sturm–Liouville case and to Weidmann [29] for the higher order case. On page 166 in [29] Weidmann states: "It may be expected that: if for every $\lambda \in (\mu_1, \mu_2)$ there exist 'sufficiently many' L^2-solutions of $(M - \lambda)u = 0$, then (μ_1, μ_2) contains no points of the essential spectrum." And in [29] this is 'almost' proven for the case when the deficiency index d is minimal. This result was recently extended by Sun, Wang and Zettl [27] to the general deficiency index case. These authors also made the following conjecture in [27] (for $k > 1$):

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Conjecture 1. *Let $M = M_Q$, $Q \in Z_n(J, \mathbb{R})$, $n = 2k$, $k \geqslant 1$, be a symmetric differential expression, $w \in L_{loc}(\mathbb{R})$, $w > 0$ on J, and let the endpoint a of J be regular. Let d be the deficiency index of (M, w) and assume that the equation has d linearly independent solutions which lie in H for every λ in an open interval I of the real line. Then there is no essential spectrum in I for any self-adjoint realization S of $S_{\min}$.*

Here $M = M_Q$ is a very general symmetric differential expression of even order with real-valued coefficients. (See Section 2 below for the definition of M.)

In this paper we prove this conjecture, also for the important case $k = 1$, under a mild additional hypothesis. As in [27] and [29] we construct operators S_t, acting in the Hilbert space $H_t = L^2((a,t), w)$, which converge in an appropriate sense to a given operator S in H as $t \to b$. Such a construction was also used [29] for the minimal deficiency case where there is no singular boundary condition. For all other values of d there are singular boundary conditions present. To overcome the formidable obstacles posed by these, we use the construction of singular boundary conditions from [26], which is based on the characterization of self-adjoint domains in [24,25]. A key feature of this characterization is the construction of LC and LP solutions. These solutions determine which of the boundary conditions of S_t are 'inherited' from those of S and which ones are not. This determination plays a critical role in the limit $S_t \to S$.

2. Statement of the main results

We study spectral properties of the self-adjoint realizations of the equation

$$My = \lambda w y \quad \text{on } J = (a,b), \quad -\infty < a < b \leqslant \infty \tag{2.1}$$

in the Hilbert space $H = L^2(J, w)$, where M is a general symmetric quasi-differential expression of order $n = 2k$, $k \geqslant 1$, with real-valued coefficients, $w \in L_{loc}(J)$, $w > 0$ on J, the endpoint a is regular and the endpoint b is singular. The case when b is regular and a is singular is entirely similar and therefore will not be explicitly stated separately.

For sufficiently smooth real-valued coefficients, the most general symmetric (formally self-adjoint) differential expressions of order $n = 2k$, $k \geqslant 1$, have the form [6,21],

$$My = \sum_{j=0}^{k} \left(p_j y^{(j)}\right)^{(j)}. \tag{2.2}$$

We are interested in using much weaker conditions, i.e., local Lebesgue integrability, on the coefficients. For this purpose Eq. (2.2) is modified by using quasi-derivatives $y^{[j]}$ as follows:

For $J = (a,b)$ an interval with $-\infty < a < b \leqslant \infty$ and $n = 2k$, $k \geqslant 1$, let

$$\begin{aligned} Z_n(J,R) := \big\{ Q = (q_{rs})_{r,s=1}^n,\ & q_{rs} \text{ real-valued} \\ & q_{r,r+1} \neq 0 \text{ a.e. on } J,\ q_{r,r+1}^{-1} \in L_{loc}(J),\ 1 \leqslant r \leqslant n-1, \\ & q_{rs} = 0 \text{ a.e. on } J,\ 2 \leqslant r+1 < s \leqslant n; \\ & q_{rs} \in L_{loc}(J),\ s \neq r+1,\ 1 \leqslant r \leqslant n-1 \big\}. \end{aligned} \tag{2.3}$$

For $Q \in Z_n(J, \mathbb{R})$ we define $V_0 := \{y\colon\ J \to \mathbb{C},\ y \text{ is measurable}\}$ and

$$y^{[0]} := y \quad (y \in V_0). \tag{2.4}$$

Inductively, for $r = 1, \dots, n$, we define

$$\begin{aligned} & V_r = \left\{ y \in V_{r-1}\colon\ y^{[r-1]} \in \left(AC_{loc}(J)\right) \right\}, \\ & y^{[r]} = q_{r,r+1}^{-1} \left\{ y^{[r-1]\prime} - \sum_{s=1}^{r} q_{rs} y^{[s-1]} \right\} \quad (y \in V_r), \end{aligned} \tag{2.5}$$

where $q_{n,n+1} := 1$, and $AC_{loc}(J)$ denotes the set of complex-valued functions which are absolutely continuous on all compact subintervals of J. Finally we set

$$My = M_Q y := (-1)^k y^{[n]} \quad (y \in V_n). \tag{2.6}$$

The expression $M = M_Q$ is called the quasi-differential expression associated with Q. For V_n we also use the notations $V(M)$ and $D(Q)$. The vector function $y^{[r]}$ $(0 \leqslant r \leqslant n)$ is called the rth quasi-derivative of y. Since the quasi-derivative depends on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$.

We now define *symmetric* quasi-differential expressions Q, these generate symmetric and self-adjoint operators in the Hilbert space H.