J. Math. Anal. Appl. 377 (2011) 834–840

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Multiplicity of solutions for a class of semilinear Schrödinger equations with sign-changing potential

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ARTICLE INFO

Article history:
Received 18 September 2010
Available online 2 December 2010
Submitted by Goong Chen

Keywords:
Schrödinger equation
Superquadratic
Sign-changing potential

ABSTRACT

In this paper, we study the existence of infinitely many nontrivial solutions for a class of semilinear Schrödinger equations $-\Delta u + V(x)u = f(x,u)$, $x \in \mathbb{R}^N$, where the primitive of the nonlinearity f is of superquadratic growth near infinity in u and the potential V is allowed to be sign-changing.

1. Introduction and main results

We consider the following semilinear Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.1}$$

With the aid of variational methods, the existence and multiplicity of nontrivial solutions for problem (1.1) have been extensively investigated in the literature over the past several decades. Many papers deal with the autonomous case where the potential V and the nonlinearity f are independent of x, or with the radially symmetric case where V and f depend on $|x|$. We quote here [1,12,13], where the autonomous case is studied, [2,3,9], where the radial nonautonomous case is considered. If the radial symmetry is lost, the problem becomes very different because of the lack of compactness. Ever since the work of Ding and Ni [6], Li [8] and Rabinowitz [11], this situation has been treated in a great number of papers under various growth conditions on V and f. For the case where the nonlinearity f is superlinear and subcritical, Rabinowitz proved in [11] the existence of a nontrivial solution for (1.1) provided that $V(x) \to +\infty$ as $|x| \to \infty$ and f satisfies the so called Ambrosetti–Rabinowitz superquadratic condition. Later, with the symmetry assumption, Bartsch and Wang obtained in [4] the existence of infinitely many solutions for (1.1) under a somewhat weaker condition on V (see (b_2) in [4]) and the same conditions on f as in [11]. In [15], under the same conditions on V as in [4], Zou used the variant fountain theorem established there to obtain the same result without the Ambrosetti–Rabinowitz superquadratic condition on f, but there the primitive of f must be of μ-order ($\mu > 2$) growth near infinity in u and some monotonicity condition is required on $f(x,u)/|u|$.

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1 Partially supported by CPSF (20060400698) and the Key Foundation of Xuzhou Normal University (058LA03).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2010.11.059

In the present paper, we will study the existence of infinitely many nontrivial solutions of (1.1) under the assumptions that V satisfies some weaker conditions than those in [4] and the primitive of f satisfies a more general superquadratic condition near infinity. Precisely, we require the following conditions on V.

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ is bounded from below.
(V_2) There exists $r_0 > 0$ such that

$$\lim_{|y|\to\infty} meas\big(\{x \in \mathbb{R}^N\colon |x-y| \leqslant r_0,\ V(x) \leqslant M\}\big) = 0, \quad \forall M > 0,$$

where $meas(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^N$.

Concerning the nonlinearity f and its primitive $F(x,u) := \int_0^u f(x,s)\,ds$, we make the following assumptions:

(S_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $c_1 > 0$ and $2 < \nu < 2^*$ such that

$$\big|f(x,u)\big| \leqslant c_1\big(|u| + |u|^{\nu-1}\big), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where 2^* denotes the critical Sobolev exponent, i.e., $2^* = 2N/(N-2)$ for $N \geqslant 3$ and $2^* = \infty$ for $N = 1, 2$.
(S_2) $F(x,0) \equiv 0$, $F(x,u) \geqslant 0$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, and $\lim_{|u|\to\infty} F(x,u)/u^2 \to \infty$ uniformly on $\mathbb{R}^N$.
(S_3) There exists a constant $\vartheta \geqslant 1$ such that

$$\vartheta \widetilde{F}(x,u) \geqslant \widetilde{F}(x,su), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R} \text{ and } s \in [0,1],$$

where $\widetilde{F}(x,u) := uf(x,u) - 2F(x,u)$.
(S_4) $f(x,-u) = -f(x,u)$, $\forall (x,u) \in \mathbb{R}^N \times \mathbb{R}$.

Our main result reads as follows.

Theorem 1.1. *Suppose that* (V_1), (V_2) *and* (S_1)–(S_4) *are satisfied. Then problem* (1.1) *possesses infinitely many nontrivial solutions.*

Remark 1.2. Conditions like (V_1) and (V_2) have been given in [5], but there $\inf_{\mathbb{R}^N} V(x) > 0$ is required. As shown in [10], the condition (S_3) due to [7] is somewhat weaker than the condition that $f(x,u)/|u|$ is nondecreasing in u for all $x \in \mathbb{R}^N$. Besides, we note that the usual condition $\lim_{u\to 0} f(x,u)/u = 0$ is not needed in our Theorem 1.1. Let V be a zig–zag function with respect to $|x|$ defined by

$$V(x) = \begin{cases} 2n|x| - 2n(n-1) + c_0, & n-1 \leqslant |x| < (2n-1)/2, \\ -2n|x| + 2n^2 + c_0, & (2n-1)/2 \leqslant |x| \leqslant n, \end{cases} \quad n \in \mathbb{N} \text{ and } c_0 \in \mathbb{R}$$

and

$$f(x,u) = a(x)u\ln\big(2 + |u|\big), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where a is a continuous bounded function with positive lower bound. It's easy to check that V and f satisfy (V_1), (V_2) and (S_1)–(S_4) in our Theorem 1.1, but V does not satisfy the condition (b_2) in [4] and the primitive of f neither satisfies the Ambrosetti–Rabinowitz superquadratic condition nor is of μ-order ($\mu > 2$) growth near infinity in u.

2. Variational setting and proof of the main result

Before establishing the variational setting for our problem (1.1), we have the following:

Remark 2.1. From (V_1), we know that there exists a constant $V_0 > 0$ such that $\overline{V}(x) := V(x) + V_0 \geqslant 1$ for all $x \in \mathbb{R}^N$. Let $\bar{f}(x,u) = f(x,u) + V_0 u$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ and consider the following new Schrödinger equation

$$\begin{cases} -\Delta u + \overline{V}(x)u = \bar{f}(x,u), & x \in \mathbb{R}^N, \\ u \in H^1\big(\mathbb{R}^N\big). \end{cases} \tag{2.1}$$

Then problem (2.1) is equivalent to problem (1.1). It's easy to check that the hypotheses (V_1), (V_2) and (S_1)–(S_4) still hold for $\overline{V}$ and $\bar{f}$ provided that those hold for V and f. Hence we can assume without loss of generality that $V(x) \geqslant 1$ for all $x \in \mathbb{R}^N$ in (V_1).

In view of Remark 2.1, we consider the space $E := \{u \in H^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)u^2\,dx < \infty\}$ equipped with the following inner product

$$(u,v) = \int_{\mathbb{R}^N} \big(\nabla u \cdot \nabla v + V(x)uv\big)\,dx.$$