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Bijections and symmetries for the factorizations of the long cycle

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ABSTRACT

We study the factorizations of the permutation $(1,2,\ldots,n)$ into k factors of given cycle types. Using representation theory, Jackson obtained for each k an elegant formula for counting these factorizations according to the number of cycles of each factor. In the cases $k=2,3$ Schaeffer and Vassilieva gave a combinatorial proof of Jackson's formula, and Morales and Vassilieva obtained more refined formulas exhibiting a surprising symmetry property. These counting results are indicative of a rich combinatorial theory which has remained elusive to this point, and it is the goal of this article to establish a series of bijections which unveil some of the combinatorial properties of the factorizations of $(1,2,\ldots,n)$ into k factors for all k. We thereby obtain refinements of Jackson's formulas which extend the cases $k=2,3$ treated by Morales and Vassilieva. Our bijections are described in terms of "constellations", which are graphs embedded in surfaces encoding the transitive factorizations of permutations.

1. Introduction

We consider the problem of enumerating the factorizations of the permutation $(1,2,\ldots,n)$ into k factors according to the cycle type of each factor. In [6] Jackson established a remarkable *counting formula* (analogous to the celebrated Harer–Zagier formula [5]) characterizing the generating function of the factorizations of the long cycle according to the number of cycles of each factor. A combinatorial proof was subsequently given for the cases $k=2,3$ by Schaeffer and Vassilieva [14,13]. Building on these bijections, Morales and Vassilieva also established for $k=2,3$ a formula for the generating

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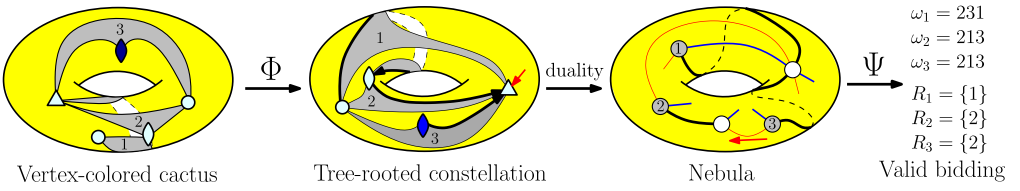

Fig. 1. Summary of the bijections presented in this article. We start with the classical encoding of the factorizations of $(1,2,\ldots,n)$ by cacti (here $k=n=3$). By this encoding, the objects to be enumerated are *vertex-colored cacti*. We first establish a bijection Φ between vertex-colored cacti and *tree-rooted constellations* (Section 3). We then apply duality to tree-rooted constellations. By characterizing the dual of tree-rooted constellations we obtain a correspondence with *nebulas* (Section 5). Lastly we establish a bijection Ψ between nebulas and *valid biddings* (Section 6).

function of the factorizations of $(1,2,\ldots,n)$ counted according to the cycle type of each factor [11, 12]. These formulas display a surprising *symmetry property* which has remained unexplained so far.

In this article we explore the combinatorics of the factorizations of the permutation $(1,2,\ldots,n)$ through a series of bijections. All our bijections are described in terms of *maps* and *constellations* which are graphs embedded in surfaces encoding the transitive factorizations in the symmetric group (see Section 2 for definitions). A summary of our bijections is illustrated in Fig. 1. Our first bijection gives an encoding of the factorizations of the permutation $(1,2,\ldots,n)$ into *tree-rooted k-constellations*. This encoding allows one to easily prove the case $k=2$ of Jackson's counting formula, as well as to establish the symmetry property for all $k\geqslant 2$. However for $k\geqslant 3$, the tree-rooted k-constellations are still difficult to count and we give further bijections. Eventually, we show bijectively that proving Jackson's counting formula reduces to proving an intriguing probabilistic statement (see Theorem 1.7). In Section 7 we prove this probabilistic statement in the cases $k=2,3,4$ (thereby proving Jackson's counting formula for these cases) but the cases $k>4$ shall be treated (along with similar probabilistic statements) in a separate paper [3]. Before describing our results further we need to review the literature.

Enumerative results about the factorizations of the long cycle. Given k partitions $\lambda^{(1)},\ldots,\lambda^{(k)}$ of n, it is a classical problem to determine the number $\kappa(\lambda^{(1)},\ldots,\lambda^{(k)})$ of factorizations $\pi_1\circ\pi_2\circ\cdots\circ\pi_k=(1,2,\ldots,n)$ such that the permutation π_t has cycle type $\lambda^{(t)}$ for all $t\in[k]:=\{1,\ldots,k\}$. By the general theory of group representations, the *connection coefficients* $\kappa(\lambda^{(1)},\ldots,\lambda^{(k)})$ can be expressed in terms of the characters of the symmetric group, but this expression is not really explicit even for $k=2$. However, Jackson established in [6] a remarkable formula for the generating function of factorizations counted according to the number of cycles of the factors.

Definition 1.1. We define $\mathcal{M}^n_{p_1,\ldots,p_k}$ as the set of n-tuples $(R_1,\ldots,R_n)$ of proper subsets R_t of $[k]$ such that each integer $t\in[k]$ appears in exactly p_t of the subsets $R_1,\ldots,R_n$. We denote by $M^n_{p_1,\ldots,p_k}$ the cardinality of $\mathcal{M}^n_{p_1,\ldots,p_k}$, which is clearly equal to the coefficient of $x_1^{p_1}\cdots x_k^{p_k}$ in the polynomial $(\prod_{i=1}^k(1+x_i)-\prod_{i=1}^k x_i)^n$.

Jackson's formula is

$$\sum_{\pi_1\circ\cdots\circ\pi_k=(1,2,\ldots,n)}\prod_{i=1}^k x_i^{\ell(\pi_i)}=n!^{k-1}\sum_{p_1,\ldots,p_k>0}\left(\prod_{i=1}^k\binom{x_i}{p_i}\right)M^{n-1}_{p_1-1,\ldots,p_k-1},\tag{1.1}$$

where $\ell(\pi)$ is the number of cycles of the permutation π, and $\binom{x}{p}=\frac{x(x-1)\cdots(x-p+1)}{k!}$. Observe that the sum in the right-hand side of (1.1) is finite since $M^{n-1}_{p_1-1,\ldots,p_k-1}=0$ whenever $p_t>n$.

Jackson's formula can equivalently be stated in terms of *colored factorizations*.