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Sparsity optimized high order finite element functions for $H(\text{curl})$ on tetrahedra

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ABSTRACT

$H(\text{curl})$ conforming finite element discretizations are a powerful tool for the numerical solution of the system of Maxwell's equations in electrodynamics. In this paper we construct a basis for conforming high-order finite element discretizations of the function space $H(\text{curl})$ in 3 dimensions. We introduce a set of hierarchic basis functions on tetrahedra with the property that both the L^2-inner product and the $H(\text{curl})$-inner product are sparse with respect to the polynomial degree. The construction relies on a tensor-product based structure with properly weighted Jacobi polynomials as well as an explicit splitting of the basis functions into gradient and non-gradient functions. The basis functions yield a sparse system matrix with $\mathcal{O}(1)$ nonzero entries per row.
The proof of the sparsity result on general tetrahedra defined in terms of their barycentric coordinates is carried out by an algorithm that we implemented in Mathematica. A rewriting procedure is used to explicitly evaluate the inner products. The precomputed matrix entries in this general form for the cell-based basis functions are available online.

1. Introduction

The main result of this paper is the construction of high order finite element basis functions for $H(\text{curl})$ on tetrahedra yielding a sparse system matrix. These basis functions are defined via certain

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Jacobi-type polynomials. Proving the sparsity, integrals over products of these basis functions and their partial derivatives need to be evaluated. Even though Jacobi polynomials have been studied extensively in the past, for this evaluation several identities are needed that are not yet folklore in the literature. For obtaining these necessary relations we invoke recently developed computer algebra algorithms. Furthermore the amount of data that needs to be handled forbids classical hand computations and we employ a program implemented in Mathematica to carry out this task.

The symbolic component in the construction of the basis functions and the proof of their properties is the main focus of the present work. The gain of invoking symbolic computation is twofold: on the one hand it is used as a practical tool to derive necessary identities and relations, on the other hand it is inevitable for dealing with the large number of integrals to be evaluated in a systematic manner. For the proof of the main result we use the packages "HolonomicFunctions" [27] and "SumCracker" [26] that are explained in more detail below. These are among several available tools for dealing with special functions in a symbolic way, such as, e.g., [45,17,16,35,36,43]. The proof of the main result proceeds by a rewrite procedure of the given integrals that relies on identities discovered using these packages.

Finite element methods are nowadays the preferred tool for numerically solving partial differential equations (PDEs) on complicated domains, see e.g. [33,13]. In the presence of smooth solutions the convergence rate of this approximation procedure can be accelerated significantly if basis functions of high polynomial degrees are used. This is called the p- and hp-version of the FEM, see e.g. [38,19,5]. The implementation of these methods however then becomes very involved and every simplification is most welcome [11,42,24,21,37].

This note is the last in a series of papers dealing with the construction of sparsity optimized basis functions for different Sobolev spaces [10,9,6,8,7]. Except for the first one [8] that dealt with basis functions defined on triangles only, the computations were handed over to a computer algebra system. Still, the focus of these papers was on the numerical aspects of the construction.

$H(\mathrm{curl})$ conforming basis functions are chosen to be piecewise polynomial functions on tetrahedrons with globally continuous tangential components along the interfaces of the tetrahedrons, see [11,31,42,21]. The construction of the basis functions for the vector valued space $H(\mathrm{curl})$ follows the approach presented by Zaglmayr [37,44]. They are built starting from (in principle) any set of H^1-conforming, i.e. globally continuous, basis functions and they are divided into curl-free basis functions and a set of non-curl-free basis functions that complete the basis. As we show below, they yield sparse system matrices and this is of advantage in the numerical computation concerning both computing time and memory requirement.

The outline of the paper is as follows. Section 2 gives an overview about the mathematical background from partial differential equations which is required to motivate the following sections. Namely, the Maxwell equations and FEM are described very briefly. Finally, the importance of the sparsity of the system matrix is motivated. The basis functions are defined in Section 3. The main results are also formulated in this part of the paper. Section 4 summarizes the most important properties of Jacobi polynomials needed.

For the proof of the sparsity properties of the basis functions multi-integrals over certain Jacobi polynomials and weights over general tetrahedra have to be computed. We are evaluating these integrals symbolically using a rewrite procedure that we implemented in Mathematica and that is described in Section 5.

2. Maxwell's equations and the finite element method

Variational formulation and the function space $H(\mathrm{curl})$. In this paper, we investigate the following problem in variational formulation: Given μ, κ, f, find $u \in H(\mathrm{curl}, \Omega)$ such that

$$a(u, v) := \int_{\Omega} \mu^{-1} \operatorname{curl} u \cdot \operatorname{curl} v + \int_{\Omega} \kappa u \cdot v = \int_{\Omega} f \cdot v =: F(v) \quad \forall v \in H(\mathrm{curl}, \Omega) \tag{2.1}$$