Advances in Applied Mathematics 46 (2011) 192–208

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

# Eisenstein series and elliptic functions on $\Gamma_0(10)$

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ARTICLE INFO

*Article history:*
Available online 20 October 2010

Dedicated to Dennis Stanton on the occasion of his 60th birthday

*MSC:*
primary 11F11
secondary 33E05

*Keywords:*
Dedekind's eta-function
Eisenstein series
Elliptic function
Modular form
Ramanujan's second notebook
Theta function

ABSTRACT

We generalize two identities involving Eisenstein series given in Chapter 19 of Ramanujan's second notebook. Four infinite families of Eisenstein series are obtained and their properties are investigated. The generating function of each infinite family is shown to be an elliptic function which has a simple infinite product expansion. Transformation properties of the Eisenstein series are investigated from first principles without using the transformation properties of Dedekind's eta-function.



## 1. Introduction and definitions

Let $\tau$ be a complex number that satisfies $\operatorname{Im}\tau > 0$ and let $q = \exp(2\pi i\tau)$. In his second notebook [12, Ch. 19, Entry 8], S. Ramanujan recorded the identities

$$1+\sum_{j=1}^{\infty}(-1)^j\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j}=\frac{1}{4}\varphi(-q)\varphi\left(-q^5\right)\left(5\varphi^2\left(-q^5\right)-\varphi^2(-q)\right) \tag{1}$$

and

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0196-8858/$ – see front matter 
doi:10.1016/j.aam.2009.12.007

$$\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^{2j}} = q\psi(q)\psi\left(q^5\right)\left(\psi^2(q)-5q\psi^2\left(q^5\right)\right), \tag{2}$$

where $(\frac{j}{5})$ is the Legendre symbol, and $\varphi$ and $\psi$ are Ramanujan's theta functions defined by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{and} \quad \psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}.$$

Proofs of (1) and (2) have been given by B.C. Berndt [2, p. 249]. On the other hand, Ramanujan knew explicit formulas for each of $\varphi(q)$, $\psi(q)$, $5\varphi^2(q^5)-\varphi^2(q)$ and $\psi^2(q)-5q\psi^2(q^5)$ as infinite products [12, Ch. 16, Entry 22], [13, p. 56] (see also [1, p. 30, (i)–(iv)] and [2, p. 36]), and from these it follows that

$$1+\sum_{j=1}^{\infty}(-1)^j\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j} = \prod_{j=1}^{\infty}\frac{(1-q^j)(1-q^{2j})^2(1-q^{5j})^3}{(1-q^{10j})^2} \tag{3}$$

and

$$\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^{2j}} = q\prod_{j=1}^{\infty}\frac{(1-q^j)^2(1-q^{2j})(1-q^{10j})^3}{(1-q^{5j})^2}. \tag{4}$$

Let $c(j)=(-1)^j(\frac{j}{5})$ and define $k$ and $z$ by

$$k = q\prod_{j=1}^{\infty}\left(1-q^j\right)^{-c(j)} \quad \text{and} \quad z = q\frac{d}{dq}\log k. \tag{5}$$

The parameter $k$ is related to the Rogers–Ramanujan continued fraction. This connection was noted by Ramanujan in his second notebook [12, p. 326] where $k$ was denoted by $n$ (see also [3, p. 13]). Several further identities involving $k$ are stated without proof in Ramanujan's lost notebook [13, pp. 53, 56, 208]. Several of Ramanujan's identities for $k$ were discussed by S. Raghavan and S.S. Rangachari [11]. Ramanujan's results were analyzed in detail by S.-Y. Kang [9]. Kang's proofs rely on modular equations in Ramanujan's notebooks [12] that have been proved by B.C. Berndt [2]. Her work has been reproduced in the book by G.E. Andrews and B.C. Berndt [1, pp. 33–44, 81–84]. For different proofs, see the work of C. Gugg [8]. In [7], Ramanujan's results were extended and the parameter $z$ was introduced.

Clearly, from the definition (5),

$$1+\sum_{j=1}^{\infty}(-1)^j\left(\frac{j}{5}\right)\frac{jq^j}{1-q^j} = z, \tag{6}$$

and it was shown in [7] that

$$\sum_{j=1}^{\infty}\left(\frac{j}{5}\right)\frac{jq^j}{1-q^{2j}} = \frac{zk}{1-k^2}. \tag{7}$$

Moreover, combining (3)–(7) it follows that $z$ and $zk/(1-k^2)$ each have expressions as infinite products.