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Cellular spanning trees and Laplacians of cubical complexes

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A B S T R A C T

We prove a Matrix-Tree Theorem enumerating the spanning trees of a cell complex in terms of the eigenvalues of its cellular Laplacian operators, generalizing a previous result for simplicial complexes. As an application, we obtain explicit formulas for spanning tree enumerators and Laplacian eigenvalues of cubes; the latter are integers. We prove a weighted version of the eigenvalue formula, providing evidence for a conjecture on weighted enumeration of cubical spanning trees. We introduce a cubical analogue of shiftedness, and obtain a recursive formula for the Laplacian eigenvalues of shifted cubical complexes, in particular, these eigenvalues are also integers. Finally, we recover Adin's enumeration of spanning trees of a complete colorful simplicial complex from the Cellular Matrix-Tree Theorem together with a result of Kook, Reiner and Stanton.

"Is there a q-analogue of that?"
Dennis Stanton

1. Introduction

1.1. Cellular spanning trees

In [10], the authors initiated the study of *simplicial spanning trees*: subcomplexes of a simplicial complex that behave much like the spanning trees of a graph. The central result of [10] is a general-

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ization of the Matrix-Tree Theorem, enumerating simplicial spanning trees in terms of eigenvalues of combinatorial Laplacians. In this paper, we extend our field of inquiry to the setting of arbitrary cell complexes and their Laplacians.

Let X be a d-dimensional cell complex; we write X_i for the set of i-cells of X. A *cellular spanning tree*, or just a *spanning tree*, of X is a d-dimensional subcomplex $Y \subseteq X$ with the same $(d-1)$-skeleton, whose (reduced) homology satisfies the two conditions $\tilde{H}_d(Y;\mathbb{Z}) = 0$ and $|\tilde{H}_{d-1}(Y;\mathbb{Z})| < \infty$. These two conditions imply that $|Y_d| = |X_d| - \tilde{\beta}_d(X) + \tilde{\beta}_{d-1}(X)$, where $\tilde{\beta}_d$ and $\tilde{\beta}_{d-1}$ denote (reduced) Betti numbers. In fact, any two of these three conditions together imply the third. In the case $d = 1$, a cellular spanning tree is just a spanning tree of X in the usual graph-theoretic sense; the first two conditions above say respectively that Y is acyclic and connected, and the third condition says that Y has one fewer edge than the number of vertices in X.

Let C_i denote the i-th cellular chain group of X with coefficients in $\mathbb{Z}$, and let $\partial_i : C_i \to C_{i-1}$ and $\partial_i^* : C_{i-1} \to C_i$ be the cellular boundary and coboundary maps (where we have identified cochains with chains via the natural inner product). The *i-th up-down, down-up and total combinatorial Laplacians* are respectively

$$L_i^{\mathrm{ud}} = \partial_{i+1}\partial_{i+1}^*, \qquad L_i^{\mathrm{du}} = \partial_i^*\partial_i, \qquad L_i^{\mathrm{tot}} = L_i^{\mathrm{ud}} + L_i^{\mathrm{du}},$$

which may be viewed either as endomorphisms on C_i, or as square symmetric matrices, as convenient. We are interested in the *spectra* of these Laplacians, that is, their multisets of eigenvalues, which we denote by $s_i^{\mathrm{ud}}(X)$, $s_i^{\mathrm{du}}(X)$, and $s_i^{\mathrm{tot}}(X)$ respectively. Combinatorial Laplacians seem to have first appeared in the work of Eckmann [11] on finite-dimensional Hodge theory, in which the i-th homology group of a chain complex is identified with $\ker(L_i)$ via the direct sum decomposition $C_i = \operatorname{im}\partial_{i+1} \oplus \ker L_i \oplus \operatorname{im}\partial_i^*$. As the name suggests, the combinatorial Laplacian is a discrete version of the Laplacian on differential forms for a Riemannian manifold; Dodziuk and Patodi [7] proved that for suitably nice triangulations, the eigenvalues of the combinatorial and analytic Laplacians converge to each other.

For $0 \leqslant k \leqslant d$, let $\mathcal{T}_k(X)$ denote the set of all spanning k-trees of X (that is, the spanning trees of the k-skeleton of X). Let

$$\tau_k(X) = \sum_{Y \in \mathcal{T}_k(X)} \left|\tilde{H}_{k-1}(Y;\mathbb{Z})\right|^2,$$

$$\pi_k(X) = \prod_{0 \neq \lambda \in s_{k-1}^{\mathrm{ud}}(X)} \lambda \quad \text{for } k \geqslant 1.$$

We set $\pi_0 = |X_0|$, the number of vertices of X, for reasons that will become clear later. Note that $\tau_1(X)$ is just the number of spanning trees of the 1-skeleton of X. Bolker [4] was the first to observe that enumeration of higher-dimensional trees requires some consideration of torsion; the specific summand $|\tilde{H}_{k-1}(Y;\mathbb{Z})|^2$, first noticed by Kalai [16], arises from an application of the Binet–Cauchy theorem. The precise relationship between the families of invariants $\{\pi_k(X)\}$ and $\{\tau_k(X)\}$ is as follows.

Theorem 2.8 *(Cellular Matrix-Tree Theorem). Let $d \geqslant 1$, and let X^d be a cell complex such that $H_i(X;\mathbb{Q}) = 0$ for all $i < d$. Fix a spanning $(d-1)$-tree $Y \subset X$, let $U = Y_{d-1}$, and let L_U be the matrix obtained from $L_{d-1}^{\mathrm{ud}}(X)$ by deleting the rows and columns corresponding to U. Then*:

$$\pi_d(X) = \frac{\tau_d(X)\tau_{d-1}(X)}{|\tilde{H}_{d-2}(X;\mathbb{Z})|^2} \quad \textit{and}$$

$$\tau_d(X) = \frac{|\tilde{H}_{d-2}(X;\mathbb{Z})|^2}{|\tilde{H}_{d-2}(Y;\mathbb{Z})|^2} \det L_U.$$