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On an identity of Gessel and Stanton and the new little Göllnitz identities

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ABSTRACT

We show that an identity of Gessel and Stanton [I. Gessel, D. Stanton, Applications of q-Lagrange inversion to basic hypergeometric series, Trans. Amer. Math. Soc. 277 (1983) 197, Eq. (7.24)] can be viewed as a symmetric version of a recent analytic variation of the little Göllnitz identities. This is significant, since the Göllnitz–Gordon identities are considered the usual symmetric counterpart to little Göllnitz theorems. Is it possible, then, that the Gessel–Stanton identity is part of an infinite family of identities like those of Göllnitz–Gordon?

Toward this end, we derive partners and generalizations of the Gessel–Stanton identity. We show that the new little Göllnitz identities enumerate partitions into distinct parts in which even-indexed (resp. odd-indexed) parts are even, and derive a refinement of the Gessel–Stanton identity that suggests a similar interpretation is possible. We study an associated system of q-difference equations to show that the Gessel–Stanton identity and its partner are actually two members of a three-element family.

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1. Introduction

In 1983, Gessel and Stanton presented the following Rogers–Ramanujan type identity [11, p. 197, Eq. (7.24)]:

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{2n} q^{2n^2}}{(q^8;q^8)_n (q^2;q^4)_n} = \left(-q^3,-q^5;q^8\right)_\infty \left(-q^2;q^2\right)_\infty, \tag{1.1}$$

where

$$(a;q)_\infty := \prod_{j=0}^{\infty} \left(1-aq^j\right),$$

$$(a_1,a_2,\ldots,a_r;q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_r;q)_\infty,$$

and

$$(a;q)_n := (a)_\infty / \left(aq^n\right)_\infty.$$

If the base in a rising q-factorial is omitted, it is assumed to be q, i.e.

$$(a)_\infty := (a;q)_\infty \quad \text{and} \quad (a)_n := (a;q)_n.$$

Rogers–Ramanujan type identities rarely occur in isolation; where there is one, there is often one or more "partners" and these identities generalize in a number of ways. We will use a variety of techniques to derive partners and generalizations.

Identity (1.1) was one of many derived by Gessel and Stanton in their 1983 paper on q-Lagrange inversion [11]. This identity came to our attention as a candidate for a symmetric version of a recently discovered variation of the little Göllnitz identities [9] in the same way that the Göllnitz–Gordon theorem is a symmetric version of the familiar little Göllnitz identities.

As such, there were natural questions to ask about the Gessel–Stanton identity (1.1), such as: what would its partner(s) be? Does it have a combinatorial interpretation that relates it to the little Göllnitz identities as do the Göllnitz–Gordon identities? Does it have a multiparameter generalization? And what do those parameters count? Does the identity generalize to an infinite family as the Göllnitz–Gordon identities do?

In this paper we answer some of these questions and suggest approaches to others.

Section 2 supplies the background on the Göllnitz–Gordon theorem and its relationship to the little Göllnitz identities. We describe the "new" little Göllnitz identities, how they are related to the Gessel–Stanton identity (1.1), and why one would expect a partner for (1.1).

In Section 3, we derive a 3-parameter generalization of (1.1) and a partner for (1.1) from Andrews' q-analog of Bailey's sum [3, p. 526, Eq. (1.9)]. The new little Göllnitz identities have a similar generalization and there is a combinatorial interpretation of the generalized infinite products that encompasses both pairs.

In Section 4, the focus is on the infinite sums. We prove an interpretation of the new little Göllnitz identities, which in itself is interesting, but which we conjecture can extend to the Gessel–Stanton pair.

In Section 5 we describe the generalized Göllnitz–Gordon identities and consider whether there is an analogous generalization of the Gessel–Stanton identities to an infinite family. We study associated q-difference equations via the methods Andrews used to derive infinite families for the Rogers–Ramanujan identities and the Göllnitz–Gordon identities [5, Chapter 7]. Although we are not successful at finding the combinatorial counterpart of the analogous infinite family, we show via this