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Orthogonal sets of Young symmetrizers ☆

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ABSTRACT

Young symmetrizers are primitive idempotents in the group algebra of the symmetric group S_n that are indexed in a natural way by Young tableaux. Although the Young symmetrizers corresponding to standard tableaux may be used to decompose the group algebra into a direct sum of minimal left ideals, they are not pairwise orthogonal in general. We pose the problem of finding maximum sets of pairwise orthogonal (but not necessarily standard) Young symmetrizers, and show in particular that it is possible to find (nonstandard) complete orthogonal sets for all partitions of n if and only if $n \leqslant 6$.

1. Young symmetrizers

Let $[n] = \{1, 2, \dots, n\}$ and let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n; i.e., a weakly decreasing sequence of positive integers with sum n. The Young diagram associated to λ is the set of n lattice points $D_\lambda := \{(i, j) \in \mathbb{Z}^2\colon\ 1 \leqslant j \leqslant \lambda_i\}$, and a Young tableau of shape λ is a bijection $T : D_\lambda \to [n]$. We use matrix-style coordinates so that for example,

$$\begin{matrix} 6 & 3 & 4 & 2 \\ 1 & 8 & & \\ 7 & 5 & & \end{matrix}$$

is a Young tableau of shape $(4, 2, 2)$, and the $(3, 1)$-entry of this tableau is 7.

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Given a Young tableau T, let $R(T)$ denote the row-stabilizer of T; i.e., the subgroup of S_n that preserves the sets of entries in each row. We similarly let $C(T)$ denote the column stabilizer. In the group algebra $\mathbb{Q}S_n$, let

$$r_T := \sum_{w \in R(T)} w, \qquad \bar{c}_T := \sum_{w \in C(T)} \operatorname{sgn}(w) w.$$

The product $e_T := \bar{c}_T r_T$ is a *Young symmetrizer*; it generates a minimal left ideal in the group algebra, and the irreducible representation of S_n this ideal affords depends only on the shape λ. Conversely, every irreducible representation of S_n arises this way.

Young symmetrizers are also known to be quasi-idempotent in the sense that

$$e_T^2 = \alpha_\lambda e_T \tag{1}$$

for some nonzero scalar α_λ. In fact, $\alpha_\lambda = n!/f^\lambda$, where f^λ denotes the dimension of the corresponding representation of S_n. (For example, see Theorem 3.1.10 of [2].)

Given two Young tableaux S and T, we say that S is *rc-transverse* to T if every row of S intersects every column of T in at most one element. Dually, we say that T is *cr-transverse* to S. Note that T is rc-transverse and cr-transverse to itself.

Proposition 1. *If S and T are Young tableaux of shapes λ and μ (respectively), then $e_S e_T = 0$ if and only if $\lambda \neq \mu$ or S is not rc-transverse to T.*

Proof. If S is not rc-transverse to T, then there is a pair of elements i, j that appear in the same row of S and the same column of T. It follows that the transposition $t = (i, j)$ belongs to both $R(S)$ and $C(T)$, whence $r_S t = r_S$, $t\bar{c}_T = -\bar{c}_T$, and

$$r_S \bar{c}_T = r_S t \bar{c}_T = -r_S \bar{c}_T = 0.$$

Consequently, $e_S e_T = \bar{c}_S (r_S \bar{c}_T) r_T = 0$.

In the case $\lambda \neq \mu$, the ideal $\mathbb{Q}S_n e_S e_T$ is simultaneously a quotient of $\mathbb{Q}S_n e_S$ and a submodule of $\mathbb{Q}S_n e_T$. Since these $\mathbb{Q}S_n$-modules are irreducible and non-isomorphic, this is possible only if $e_S e_T = 0$.

For the converse, if S is rc-transverse to T and $\lambda = \mu$, then the unique permutation $w_{S,T} \in S_n$ that transforms $T \to S$ belongs to $R(S)C(T)$ (see Chapter 7, Lemma 1 in [1]). It follows that $r_S w_{S,T} \bar{c}_T = \pm r_S \bar{c}_T$, and hence (1) implies

$$\pm e_S e_T = e_S w_{S,T} e_T = w_{S,T} e_T^2 = \alpha_\lambda w_{S,T} e_T.$$

The fact that this expression is nonzero (i.e., $e_T \neq 0$) may be easily seen without taking for granted that e_T generates an irreducible $\mathbb{Q}S_n$-module. Indeed, note that $R(T)$ and $C(T)$ must intersect trivially, since any permutation common to both must stabilize the intersection of every row and column of T. From this observation, it follows that the coefficient of the identity element in $e_T = \bar{c}_T r_T$ is 1. □

We define Young tableaux S and T to be *orthogonal* if $e_S e_T = e_T s_S = 0$. By the above proposition, this is equivalent to S and T either having different shapes, or S being neither rc-transverse nor cr-transverse to T.

As noted originally by Young, a set $\mathcal{Y}$ of pairwise orthogonal Young tableaux of shape λ may be used to give an explicit set of matrix units in $\mathbb{Q}S_n$. Indeed, if we set

$$e_{S,T} := \frac{1}{\alpha_\lambda} \bar{c}_S w_{S,T} r_T = \frac{1}{\alpha_\lambda} w_{S,T} e_T = \frac{1}{\alpha_\lambda} e_S w_{S,T} \quad (S, T \in \mathcal{Y}),$$