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A unifying poset perspective on alternating sign matrices, plane partitions, Catalan objects, tournaments, and tableaux

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ABSTRACT

Alternating sign matrices (ASMs) are square matrices with entries 0, 1, or -1 whose rows and columns sum to 1 and whose nonzero entries alternate in sign. We present a unifying perspective on ASMs and other combinatorial objects by studying a certain tetrahedral poset and its subposets. We prove the order ideals of these subposets are in bijection with a variety of interesting combinatorial objects, including ASMs, totally symmetric self-complementary plane partitions (TSSCPPs), staircase shaped semistandard Young tableaux, Catalan objects, tournaments, and totally symmetric plane partitions. We prove product formulas counting these order ideals and give the rank generating function of some of the corresponding lattices of order ideals. We also prove an expansion of the tournament generating function as a sum over TSSCPPs. This result is analogous to a result of Robbins and Rumsey expanding the tournament generating function as a sum over alternating sign matrices.

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1. Background and terminology

Definition 1.1. Alternating sign matrices (ASMs) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$,
- the entries in each row and column sum to 1,
- nonzero entries in each row and column alternate in sign.

See Fig. 1 for the seven ASMs with three rows and three columns.

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$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\ \begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\ \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\ \begin{pmatrix}0&1&0\\1&-1&1\\0&1&0\end{pmatrix}\ \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\ \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\ \begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

Fig. 1. The 3×3 ASMs.

$$\begin{matrix}6&6&6&3&3&3\\6&6&6&3&3&3\\6&6&6&3&3&3\\3&3&3\\3&3&3\\3&3&3\end{matrix}\qquad\begin{matrix}6&6&6&4&3&3\\6&6&6&3&3&3\\6&6&5&3&3&2\\4&3&3&1\\3&3&3\\3&3&2\end{matrix}\qquad\begin{matrix}6&6&6&4&3&3\\6&6&6&4&3&3\\6&6&4&3&2&2\\4&4&3&2\\3&3&2\\3&3&2\end{matrix}\qquad\begin{matrix}6&6&6&5&5&3\\6&5&5&4&3&1\\6&5&4&3&2&1\\5&4&3&2&1\\5&3&2&1&1\\3&1&1\end{matrix}$$

$$\begin{matrix}6&6&6&5&5&3\\6&5&5&3&3&1\\6&5&5&3&3&1\\5&3&3&1&1\\5&3&3&1&1\\3&1&1\end{matrix}\qquad\begin{matrix}6&6&6&5&4&3\\6&6&5&3&3&2\\6&5&5&3&3&1\\5&3&3&1&1\\4&3&3&1\\3&2&1\end{matrix}\qquad\begin{matrix}6&6&6&5&4&3\\6&6&5&4&3&2\\6&5&4&3&2&1\\5&4&3&2&1\\4&3&2&1\\3&2&1\end{matrix}$$

Fig. 2. TSSCPPs inside a $6 \times 6 \times 6$ box.

In 1983 Mills, Robbins, and Rumsey conjectured that the total number of $n \times n$ alternating sign matrices is given by the expression

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{1}$$

They were unable to prove this for all n, so for the next 13 years it remained a mystery until Doron Zeilberger proved it [24]. Shortly thereafter, Greg Kuperberg gave a shorter proof using a bijection between ASMs and configurations of the statistical physics model of square ice with domain wall boundary conditions [10]. This connection with physics has since strengthened with the conjecture of Razumov and Stroganov which says that the enumeration of subclasses of ASMs gives the ground state probabilities of the dense O(1) loop model in statistical physics [15]. This conjecture has been further refined and various special cases have been proved (see for example [4]), but no general proof is known.

Expression (1) also counts totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box [1]. A *plane partition* is a two-dimensional array of positive integers which weakly decrease across rows from left to right and weakly decrease down columns. Equivalently, a plane partition π is a finite set of positive integer lattice points (i, j, k) such that if $(i, j, k) \in \pi$ and $1 \leqslant i' \leqslant i$, $1 \leqslant j' \leqslant j$, and $1 \leqslant k' \leqslant k$ then $(i', j', k') \in \pi$. One can visualize this as stacks of unit cubes pulled toward the corner of a room. A plane partition is *totally symmetric* if whenever $(i, j, k) \in \pi$ then all six permutations of (i, j, k) are also in π.

Definition 1.2. A totally symmetric self-complementary plane partition (TSSCPP) inside a $2n \times 2n \times 2n$ box is a totally symmetric plane partition which is equal to its complement in the sense that the collection of empty cubes in the box is of the same shape as the collection of cubes in the plane partition itself.

See Fig. 2 for the 7 TSSCPPs inside a $6 \times 6 \times 6$ box.

In [1] Andrews showed that TSSCPPs inside a $2n \times 2n \times 2n$ box are counted by (1). This proves that TSSCPPs inside a $2n \times 2n \times 2n$ box are equinumerous with $n \times n$ ASMs, but no explicit bijection between these two sets of objects is known. In this paper we present a new perspective which shows that these two very different sets of objects arise naturally as members of a larger class of combi-

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