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Longest increasing subsequences, Plancherel-type measure and the Hecke insertion algorithm

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ABSTRACT

We define and study the *Plancherel–Hecke probability measure* on Young diagrams; the *Hecke algorithm* of Buch–Kresch–Shimozono–Tamvakis–Yong is interpreted as a polynomial-time exact sampling algorithm for this measure. Using the results of Thomas–Yong on *jeu de taquin* for *increasing tableaux*, a symmetry property of the Hecke algorithm is proved, in terms of longest strictly increasing/decreasing subsequences of words. This parallels classical theorems of Schensted and of Knuth, respectively, on the *Schensted* and *Robinson–Schensted–Knuth* algorithms. We investigate, and conjecture about, the limit typical shape of the measure, in analogy with work of Vershik–Kerov, Logan–Shepp and others on the "longest increasing subsequence problem" for permutations. We also include a related extension of Aldous–Diaconis on *patience sorting*. Together, these results provide a new rationale for the study of increasing tableau combinatorics, distinct from the original algebraic-geometric ones concerning K-theoretic Schubert calculus.

1. Introduction and main results

1.1. Overview

Let $W_{n,q}$ denote the set of words of length n in the alphabet $\{1, 2, \ldots, q\}$. Let $\mathtt{LIS}(w)$ denote the **length of the longest strictly increasing subsequence** of $w = w_1 w_2 \cdots w_n$, i.e., the largest ℓ with a subsequence $i_1 < i_2 < \cdots < i_\ell$ such that $w_{i_1} < w_{i_2} < \cdots < w_{i_\ell}$. Similarly, we consider the **length of the longest strictly decreasing subsequence** $\mathtt{LDS}(w)$ of w. Our main goal is to introduce and study

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a discrete probability measure on Young diagrams, in connection with the study of the distributions of LIS and LDS on uniform random words. An additional goal is to provide a novel motivation for the K-theoretic Schubert calculus combinatorics of [7,24].

There are analogies with the study of LIS and LDS in the **permutation case**, i.e., when w is chosen uniformly at random from the symmetric group S_n. The latter topic has attracted considerable attention; we refer the reader to the surveys [1,23] and the references therein. In the permutation case, random Young diagrams are distributed according to the *Plancherel measure* (on irreducible representations) of S_n. This discrete probability measure is the push-forward of the uniform distribution on S_n, under the *Robinson–Schensted correspondence*. Schensted [20] established that this correspondence encodes LIS(w) and LDS(w) symmetrically in the shape λ associated to w. In [26,18], these ideas are applied to determine the asymptotics of the expectation of LIS over S_n (solving the old "longest increasing sequences problem"), via a study of the "limit typical shape" under the Plancherel measure.

As a continuation of this theme, we apply the Hecke (insertion) algorithm of [7] to define the Young diagram Heckeshape(w) for each $w \in W_{n,q}$; using this we define *Plancherel–Hecke measure*. Our belief that this measure should actually be worthy of analysis was initially guided by our theorem that Hecke symmetrically encodes LIS(w) and LDS(w) for $w \in W_{n,q}$, a generalization of Schensted's theorem. During the course of our investigation, we found that many other aspects of the Plancherel–Hecke measure (conjecturally) also resemble those of the Plancherel measure. This paper records these results, both theoretical and computational, as a justification for further study.

Briefly, this is how the two aforementioned measures compare. We consider the behaviour as n goes to infinity, and q grows proportionally to n^{α} for a fixed α. We conjecture that for $\alpha > \frac{1}{2}$, our measure is concentrated around the limit typical shape under Plancherel measure. This *Plancherel curve* plays an important role in [26,18]. On the other hand, for $\alpha < \frac{1}{2}$ we conjecture the measure is concentrated near the "staircase shape". In particular, a "phase transition" is suggested at $\alpha = \frac{1}{2}$. As we tune α, a symmetric deformation of the Plancherel curve occurs. In view of the above mentioned result on the Hecke algorithm, this transition phenomenon is further evidenced by computations (with contributions by O. Zeitouni) of the expectation of LIS and LDS as α varies; see Section 5 and Appendix A.

There have been earlier extensions of the permutation case to $W_{n,q}$. The limit distribution of the length of the longest *weakly* increasing/decreasing subsequence (LwIS/LwDS) on $W_{n,q}$ was found in work of [25], following the breakthrough [2] on the limit distribution of LIS on S_n. See also the more recent work [14]. However, analogous understanding of the distribution of LIS and LDS on $W_{n,q}$ appears to be less developed; see, e.g., [3,5,25] for contributions.

As a point of comparison and contrast with our approach, previous work on LIS, LDS and $W_{n,q}$ utilizes the combinatorics of the *Robinson–Schensted–Knuth correspondence*, which asymmetrically encodes LwIS and LDS. We offer an alternative viewpoint on the relationship between Young diagrams and LIS, LDS. New questions and conjectures are raised, stemming from the Coxeter-theoretic viewpoint of [7] (which in turn generalizes ideas of [9]).

This text expresses our desire to point out a natural link between the probabilistic combinatorics of LIS, LDS and the combinatorial algebraic geometry of K-theoretic Schubert calculus. In particular, we apply and further develop the *jeu de taquin* for *increasing tableaux* from [24], thereby giving another perspective on that work, distinct from the original one. In summary, we believe that the availability of these two disparate interpretations for [7,24] provides something atypical to recommend K-theoretic tableau combinatorics, among the large array of interesting generalizations of the classical Young tableau and symmetric function theories known today.

1.2. Plancherel–Hecke measure

We identify a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k > 0)$ with its Young diagram (in English notation); set $|\lambda| := \sum_i \lambda_i$. Let $\mathbb{Y}$ denote the set of all Young diagrams. A **filling** is an assignment of a label from $\{1, 2, \ldots, q\}$ to each box of the Young diagram λ. A filling is an **increasing tableau** if it is strictly increasing in both rows and columns. Let INC(λ, q) be the set of all increasing tableaux of shape λ.