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Clusters, generating functions and asymptotics for consecutive patterns in permutations ☆

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ABSTRACT

We use the cluster method to enumerate permutations avoiding consecutive patterns. We reprove and generalize in a unified way several known results and obtain new ones, including some patterns of lengths 4 and 5, as well as some infinite families of patterns of a given shape. By enumerating linear extensions of certain posets, we find a differential equation satisfied by the inverse of the exponential generating function counting occurrences of the pattern. We prove that for a large class of patterns, this inverse is always an entire function.

We also complete the classification of consecutive patterns of length up to 6 into equivalence classes, proving a conjecture of Nakamura. Finally, we show that the monotone pattern asymptotically dominates (in the sense that it is easiest to avoid) all non-overlapping patterns of the same length, thus proving a conjecture of Elizalde and Noy for a positive fraction of all patterns.

1. Introduction

In this paper we use the cluster method of Goulden and Jackson in order to obtain new results on the enumeration of permutations avoiding consecutive patterns. Recall that a permutation π avoids a consecutive pattern σ if no subsequence of adjacent entries of π is in the same relative order as the entries of σ. Given a pattern σ, the cluster method consists of counting partial permutations in which each element is involved in at least one occurrence of σ, the so-called clusters. By inclusion–exclusion,

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the enumeration of clusters provides the enumeration of permutations according to the number of occurrences of σ.

Counting clusters can be seen as counting linear extensions in a certain poset. For instance, if σ is the monotone pattern, the corresponding poset is simply a chain, and counting linear extensions is a trivial task. In fact, not only the monotone pattern $12\dots m$ can be analyzed in this way, but also the pattern $123\dots(s-1)(s+1)s(s+2)(s+3)\dots m$ (Corollary 2.5), and other related patterns, which we call chain patterns (Theorem 2.4). Another significant case is that of non-overlapping patterns σ, which are those for which two occurrences of σ in a permutation cannot overlap in more than one position. The associated poset is not difficult to analyze when $\sigma_1 = 1$, in which case the number of permutations avoiding $\sigma \in \mathcal{S}_m$ depends only on the value $b = \sigma_m$, and we can derive a linear differential equation (Theorem 3.1) satisfied by the inverse of the associated exponential generating function. A weaker version of this result was proved in [7] using representations of permutations as binary trees. For chain patterns and non-overlapping patterns, the differential equations that we obtain can also be deduced from the work of Khoroshkin and Shapiro [12].

A more intricate example is the pattern 1324. This case was left open in [7] and cannot be solved with the techniques from [12] either. The number of linear extensions of the associated poset is related to the Catalan numbers, and we prove that the inverse of the generating function for this pattern satisfies a linear differential equation of order five with polynomial coefficients. Again, the technique can be extended to cover the general pattern $134\dots(s+1)2(s+2)(s+3)\dots m$ (Theorem 4.2). For other patterns of length 4, namely 1423 and 2143, we find recurrence relations satisfied by their cluster numbers (Section 5), which already appeared in [3], but we are not able to find closed solutions in terms of differential equations. In fact, we conjecture that the inverse of the generating function for permutations avoiding 1423 is not D-finite. If true, this conjecture would give the first instance of a pattern with this property, and it would make a related conjecture of Noonan and Zeilberger for classical patterns less believable.

The present situation for small patterns is the following. Say that two patterns are equivalent if their numbers of occurrences in permutations have the same distribution. There are two inequivalent patterns of length 3, already solved in [7]. There are seven inequivalent patterns of length 4, four of which are solved now, but we still do not have closed solutions for 1423, 2143 and 2413. There are 25 inequivalent patterns of length 5. Four of these are easily solved with the techniques from [7], and we can now solve four additional ones, namely 12435, 12534, 13254 and 13425. The remaining 17 patterns (which include 2 non-overlapping ones) are unsolved in terms of closed solutions or differential equations. For patterns of length 6, we prove four conjectures of Nakamura [15] regarding the equivalence of certain pairs, completing the classification into equivalence classes, and proving that there are exactly 92 inequivalent patterns.

Regarding asymptotic enumeration of permutations avoiding a pattern, we prove that the monotone pattern dominates all non-overlapping patterns of the same length (Theorem 6.8), thus proving a special case of a conjecture by Elizalde and Noy [7]. It was shown by Bóna [2] that the number of non-overlapping patterns is asymptotically a positive fraction of all patterns. We also show that the inverse of the generating function of permutations according to the number of occurrences of a given pattern is an entire function in several important cases (Theorem 6.1), but not for the pattern 2143.

We conclude this introductory section with definitions and preliminaries needed in the rest of the paper. In Sections 2 and 3 we study monotone and non-overlapping patterns, and related patterns. Section 4 is devoted to the pattern 1324 and generalizations, and Section 5 to some other patterns of length 4. In Section 6 we present our asymptotic and analytic results. We end the paper with some open problems.

1.1. Consecutive patterns

Given a sequence of distinct positive integers $\tau = \tau_1 \dots \tau_k$, we define the reduction $\mathrm{st}(\tau)$ as the permutation of length k obtained by relabeling the elements of τ with $\{1, \dots, k\}$ so that the order relations among the elements remains the same. For instance $\mathrm{st}(46382) = 34251$. Given permutations $\pi \in \mathcal{S}_n$ and $\sigma \in \mathcal{S}_m$, we say that π contains σ as a consecutive pattern if $\mathrm{st}(\pi_i \dots \pi_{i+m-1}) = \sigma$ for some $i \in \{1, \dots, n-m+1\}$. We denote by $c_\sigma(\pi)$ the number of occurrences of σ in π as a consecutive