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A permanent formula for the Jones polynomial

Martin Loebl [a,*,1], Iain Moffatt [b]

[a] Department of Applied Mathematics (KAM) and Institute of Theoretical Computer Science (ITI), Charles University, Malostranske n. 25, 118 00 Praha 1, Czech Republic
[b] Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688, USA

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ABSTRACT

The permanent of a square matrix is defined in a way similar to the determinant, but without using signs. The exact computation of the permanent is hard, but there are Monte Carlo algorithms that can estimate general permanents. Given a planar diagram of a link L with n crossings, we define a $7n \times 7n$ matrix whose permanent equals the Jones polynomial of L. This result, accompanied with recent work of Freedman, Kitaev, Larsen and Wang (2003) [8], provides a Monte Carlo algorithm for any decision problem belonging to the class BQP, i.e. such that it can be computed with bounded error in polynomial time using quantum resources.

1. Introduction and statement of results

The *permanent* of an $n \times n$ matrix $A = (a_{ij})$ is defined to be

$$\mathrm{per}(A) = \sum_{\sigma \in \mathrm{Sym}_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where Sym_n is the permutation group on $\{1, \ldots, n\}$. It is well known that if A is the $(V \times W)$, 0, 1 adjacency matrix of a bipartite graph $G = (V, W, E)$, then $\mathrm{per}(A)$ is the number of perfect matchings

* Corresponding author.
E-mail addresses: loebl@kam.mff.cuni.cz (M. Loebl), imoffatt@jaguar1.usouthal.edu (I. Moffatt).
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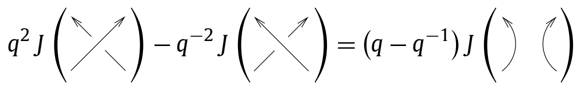
of G. The permanent of A is syntactically similar to its determinant $\det(A)$, which is a signed variation of the above sum. This mild sign variation leads to a radical change in computability: computing permanents is hard (see Section 2.2), whereas determinants can be computed in polynomial time.

It is a seminal result of Valiant (see [19]) that many graph and knot polynomials, including the Jones polynomial, may be written as permanents. Here we are interested in the expression of the Jones polynomial as a permanent and in the implications of this expression. In the case of the Jones polynomial, the general reduction of [19] leads to matrices of size at least $n^2 \times n^2$, n being the number of crossings of the link diagram. The large size of these matrices severely restricts the computational applicability of this result. It is clear that in order to efficiently calculate the Jones polynomial as a permanent, a new approach is needed. In this paper we exploit a well-known combinatorial interpretation of the permanent to find an expression of the Jones polynomial as the permanent of a matrix that grows linearly in the number of crossings.

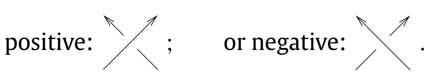
The *Jones polynomial* [13] is a celebrated invariant of links in S^3. A *link* is a disjoint union of embedded circles in 3-space. The Jones polynomial J of a link can be uniquely characterized by the following skein relation:



together with the initial condition $J(\text{unknot}) = q + q^{-1}$. From this skein relation, it follows that the Jones polynomial of a link can be computed in exponential time (with respect to the number of crossings). Although the skein relation above provides the best known definition of the Jones polynomial, there are several other ways to construct the Jones polynomial. Below we will use a statistical mechanical construction of the Jones polynomial that is due to Turaev [17] and Jones [14]. This state sum formulation for the Jones polynomial is described in Section 3.1.

In this paper we provide a permanent formula for the Jones polynomial and discuss applications and implications of this formula. This formulation of the Jones polynomial as a permanent of a $7n \times 7n$ matrix (where n is the number of crossings of the link diagram) is described below.

Consider a diagram D_L of an oriented link L, that is an oriented, 4-valent plane graph, where each vertex has a crossing structure of one of two types:



We form a graph $\hat{D}_L$, from the link diagram D_L, by replacing a neighborhood of each crossing of D_L with one of the graphs shown in Fig. 1. We say that $\hat{D}_L$ is a blown-up version of D_L. $\hat{D}_L$ is an immersed directed graph with $7n$ vertices, where n is the number of crossings of D_L. We will refer to the two graphs shown in Fig. 1, minus the incoming and outgoing edges (which come from the link diagram), as *gadgets*. We emphasize the fact that the four edges in $\hat{D}_L$ that enter and exit a gadget are all parallel when they meet the gadget.

In Definition 3.2 below, we define local weights on the edges of $\hat{D}_L$. Let M_L denote the adjacency matrix of weights of $\hat{D}_L$. M_L is a square matrix of size the number of vertices of $\hat{D}_L$, and the (i, j) entry of M_L is the weight of the corresponding directed edge (ij) of $\hat{D}_L$.

The next theorem is the main result of this paper. This result was inspired by the observation (see [9,3]) that the weight system associated with the *colored Jones function* is a permanent.

Theorem 1. *For every link L we have*

$$J(L) = q^{-2\omega(D_L)} q^{\text{rot}(L)} \text{per}(M_L).$$

Definitions of the rotation number $\text{rot}(L)$ and writhe $\omega(D_L)$ are given in Section 3.1.