Advances in Applied Mathematics 47 (2011) 772–782

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

The Tutte–Potts connection in the presence of an external magnetic field

Joanna A. Ellis-Monaghan [a,*,1], Iain Moffatt [b]

[a] Department of Mathematics, Saint Michael's College, 1 Winooski Park, Colchester, VT 05439, USA
[b] Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 36688, USA

ARTICLE INFO

Article history:
Received 22 August 2010
Revised 14 February 2011
Accepted 16 February 2011
Available online 8 April 2011

MSC:
05C31
05C22
82B20

Keywords:
Tutte polynomial
Potts model
Ising model
V-polynomial
W-polynomial
External field
Hamiltonian
Partition function
Fortuin–Kasteleyn representation
Statistical mechanics

ABSTRACT

The classical relationship between the Tutte polynomial of graph theory and the Potts model of statistical mechanics has resulted in valuable interactions between the disciplines. Unfortunately, it does not include the external magnetic fields that appear in most Potts model applications. Here we define the ***V***-polynomial, which lifts the classical relationship between the Tutte polynomial and the zero field Potts model to encompass external magnetic fields. The ***V***-polynomial generalizes Noble and Welsh's *W*-polynomial, which extends the Tutte polynomial by incorporating vertex weights and adapting contraction to accommodate them. We prove that the variable field Potts model partition function (with its many specializations) is an evaluation of the ***V***-polynomial, and hence a polynomial with deletion–contraction reduction and Fortuin–Kasteleyn type representation. This unifies an important segment of Potts model theory and brings previously successful combinatorial machinery, including complexity results, to bear on a wider range of statistical mechanics models.

* Corresponding author.
E-mail addresses: jellis-monaghan@smcvt.edu (J.A. Ellis-Monaghan), imoffatt@jaguar1.usouthal.edu (I. Moffatt).

1 The work of the first author was supported by Vermont EPSCoR.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2011.02.004

1. Introduction

The classical relationship between the Tutte polynomial, $T(G; x, y)$, and the zero-field Potts model partition function, $Z(G; q, v)$, given by

$$Z(G; q, v) = q^{k(G)} v^{|V(G)|-k(G)} T\big(G; (q+v)/v, v+1\big), \tag{1}$$

where $v = e^{\beta J} - 1$, has resulted in valuable interactions between graph theory and statistical physics. This relation assumes a zero-field Hamiltonian (Eq. (2)). However, many applications of the Potts model depend on additional terms in the Hamiltonian corresponding to the presence of additional influences (for example the standard models of magnetism, the cellular Potts model of [4], and also see [11] for examples in the life sciences). Many of these models involve edge-dependent interaction energies and site dependent external fields. The classical Tutte–Potts connection, of Eq. (1), does not apply to these situations. Here we extend the Tutte–Potts connection so that it includes the influence of external fields.

Our main result is the assimilation of the following generic form of the Hamiltonian into the theory of the Tutte–Potts connection:

$$h(\sigma) = -\sum_{\{i,j\}\in E(G)} J_{i,j}\delta(\sigma_i, \sigma_j) - \sum_{v_i\in V(G)} \sum_{\alpha=1}^{q} M_{i,\alpha}\delta(\alpha, \sigma_i),$$

where a magnetic field vector $\boldsymbol{M}_i := \{M_{i,1}, M_{i,2}, \ldots, M_{i,q}\}$ is associated to each vertex v_i. From this generic form we are able to specialize to various forms of the Hamiltonian with external fields that are common in the physics literature. To do this, we introduce the $\boldsymbol{V}$-polynomial, which is an extension of the Tutte polynomial to vertex-weighted graphs motivated by Noble and Welsh's W-polynomial from [10]. We prove that the Potts model partition function with an external field is an evaluation of the $\boldsymbol{V}$-polynomial. This gives the desired deletion–contraction reduction for the external field Potts model. The various partition functions may now be expressed as polynomials with Fortuin–Kasteleyn-type representations for them. Furthermore, this new relationship for the variable field Potts model extends the computational and analytic tools available to statistical mechanics applications. In particular, we are able to immediately transfer computational complexity results for the W- and U-polynomials to Potts model partition functions in broader settings.

2. Conventions

We assume that the reader is familiar with the connections between the Potts model and graph theory, surveyed, for example, in [1], [13] or [15]. We also assume a familiarity with Noble and Welsh's W-polynomial from [10].

2.1. The q-state Potts model

A *state* of a graph G is an assignment $\sigma : V(G) \to \{1, \ldots, q\}$, for $q \in \mathbb{Z}^+$. We let $\mathcal{S}(G)$ denote the set of states of G, and $\sigma_i := \sigma(v_i)$, for $\sigma \in \mathcal{S}(G)$. At times we will use the indices $i = 1, \ldots, n$ of the vertices in place of the vertices. We denote the interaction energy on an edge $e = \{v_i, v_j\}$ by $J_e := J_{i,j} (= J_{v_i,v_j})$, and let $j(\sigma) := -\sum_{e\in E(G)} J_e\delta(\sigma_i, \sigma_j)$. The *zero field Hamiltonian* is

$$h(\sigma) = -J \sum_{\{i,j\}\in E(G)} \delta(\sigma_i, \sigma_j), \tag{2}$$

where σ is a state of a graph G, where σ_i is the spin at vertex i, and where δ is the Kronecker delta function. To encompass various external fields and variable interaction energies, a more general form of the Hamiltonian is used.