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Minimal permutations and 2-regular skew tableaux

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A B S T R A C T

Bouvel and Pergola introduced the notion of minimal permutations in the study of the whole genome duplication-random loss model for genome rearrangements. Let $\mathcal{F}_d(n)$ denote the set of minimal permutations of length n with d descents, and let $f_d(n) = |\mathcal{F}_d(n)|$. They showed that $f_{n-2}(n) = 2^n - (n-1)n - 2$ and $f_n(2n) = C_n$, where C_n is the n-th Catalan number. Mansour and Yan proved that $f_{n+1}(2n+1) = 2^{n-2}nC_{n+1}$. In this paper, we consider the problem of counting minimal permutations in $\mathcal{F}_d(n)$ with a prescribed set of ascents, and we show that they are in one-to-one correspondence with a class of skew Young tableaux, which we call 2-regular skew tableaux. Using the determinantal formula for the number of skew Young tableaux of a given shape, we find an explicit formula for $f_{n-3}(n)$. Furthermore, by using the Knuth equivalence, we give a combinatorial interpretation of a formula for a refinement of the number $f_{n+1}(2n+1)$.

1. Introduction

The notion of minimal permutations was introduced by Bouvel and Pergola in the study of genome evolution, see [3]. Such permutations are a basis of permutations that can be obtained from the identity permutation via a given number of steps in the duplication-random loss model, see [1,3,4,6]. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation. A duplication of π means the duplication of a fragment of consecutive elements of π in such a way that the duplicated fragment is put immediately after the original fragment. Suppose that $\pi_i\pi_{i+1}\cdots\pi_j$ is the fragment for duplication, then the duplicated sequence is

$$\pi_1\cdots\pi_{i-1}\pi_i\cdots\pi_j\pi_i\cdots\pi_j\pi_{j+1}\cdots\pi_n.$$

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A random loss means to randomly delete one occurrence of each repeated element π_k for $i \leqslant k \leqslant j$, so that we get a permutation again. In the following example, the fragment 234 is duplicated, and the underlined elements are the occurrences of repeated elements that are supposed to be deleted,

$$1\,\widehat{234}\,56 \rightsquigarrow 1\,\widehat{234}\,\widehat{234}\,56 \rightsquigarrow 1\underline{2}3\underline{4}2\underline{3}456 \rightsquigarrow 132456.$$

To describe the notation of minimal permutations, we give an overview of the descent set of a permutation and the patterns of subsequences of a permutation. Let S_n be the set of permutations on $[n] = \{1, 2, \ldots, n\}$, where $n \geqslant 1$. In a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, a descent is a position i such that $i \leqslant n-1$ and $\pi_i > \pi_{i+1}$, whereas an ascent is a position i with $i \leqslant n-1$ and $\pi_i < \pi_{i+1}$. For example, the permutation $3145726 \in S_7$ has two descents 1 and 5 and has four ascents 2, 3, 4 and 6.

Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of distinct integers listed in increasing order, namely, $v_1 < v_2 < \cdots < v_n$. The standardization of a permutation π on V is the permutation $\mathrm{st}(\pi)$ on $[n]$ obtained from π by replacing v_i with i. For example, $\mathrm{st}(9425) = 4213$. A subsequence $\omega = \pi_{i(1)}\pi_{i(2)}\cdots\pi_{i(k)}$ of π is said to be of type σ or π contains a pattern σ if $\mathrm{st}(\omega) = \sigma$. We say that a permutation $\pi \in S_n$ contains a pattern $\tau \in S_k$ if there is a subsequence of π that is of type τ. For example, let $\pi = 263751498$. The subsequence 3549 is of type 1324, and so π contains the pattern 1324. We use the notation $\tau \prec \pi$ to denote that a permutation π contains the pattern τ.

A permutation π is called a minimal permutation with d descents if it is minimal in the sense that there exists no permutation σ with exactly d descents such that $\sigma \prec \pi$. Denote by $\mathcal{B}_d$ the set of minimal permutations with d descents. Bouvel and Pergola [3] have shown that the length, namely, the number of elements, of any minimal permutation in the set $\mathcal{B}_d$ is at least $d+1$ and at most $2d$. They also proved that in the whole genome duplication-random loss model, the permutations that can be obtained from the identity permutation in at most p steps can be characterized as permutations with $d = 2^p$ descents that avoid certain patterns.

Theorem 1.1 *(Bouvel and Pergola [3]). Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation on $[n]$. Then π is a minimal permutation with d descents if and only if π is a permutation with d descents satisfying the following conditions*:

(i) *It starts and ends with a descent*;
(ii) *If i is an ascent, that is, $\pi_i < \pi_{i+1}$, then $i \in \{2, 3, \ldots, n-2\}$ and $\pi_{i-1}\pi_i\pi_{i+1}\pi_{i+2}$ is of type* 2143 *or* 3142.

Denote by $\mathcal{F}_d(n)$ the set of minimal permutations of length n with d descents and $f_d(n) = |\mathcal{F}_d(n)|$. Clearly, $f_d(n) = 0$ for $d \leqslant 0$ or $d \geqslant n$, and $f_d(d+1) = 1$ for all $d \geqslant 1$. Bouvel and Pergola proved that $f_n(2n)$ equals the n-th Catalan number, that is,

$$f_n(2n) = C_n = \frac{1}{n+1}\binom{2n}{n}$$

and $f_{n-2}(n)$ is given by the formula

$$f_{n-2}(n) = 2^n - (n-1)n - 2.$$

Mansour and Yan [7] have shown that

$$f_{n+1}(2n+1) = 2^{n-2}nC_{n+1}. \tag{1.1}$$

As remarked by Bouvel and Pergola, it is an open problem to compute $f_d(n)$ for other cases of d. In this paper, we consider the enumeration of minimal permutations in $\mathcal{F}_d(n)$ with a prescribed set of ascents. We show that such minimal permutations are in one-to-one correspondence with a class of skew Young tableaux, which we call 2-regular skew tableaux. Thus we may employ the