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Spanning trees of 3-uniform hypergraphs

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ABSTRACT

Masbaum and Vaintrob's "Pfaffian matrix-tree theorem" implies that counting spanning trees of a 3-uniform hypergraph (abbreviated to 3-graph) can be done in polynomial time for a class of "3-Pfaffian" 3-graphs, comparable to and related to the class of Pfaffian graphs. We prove a complexity result for recognizing a 3-Pfaffian 3-graph and describe two large classes of 3-Pfaffian 3-graphs – one of these is given by a forbidden subgraph characterization analogous to Little's for bipartite Pfaffian graphs, and the other consists of a class of partial Steiner triple systems for which the property of being 3-Pfaffian can be reduced to the property of an associated graph being Pfaffian. We exhibit an infinite set of partial Steiner triple systems that are not 3-Pfaffian, none of which can be reduced to any other by deletion or contraction of triples.
We also find some necessary or sufficient conditions for the existence of a spanning tree of a 3-graph (much more succinct than can be obtained by the currently fastest polynomial-time algorithm of Gabow and Stallmann for finding a spanning tree) and a superexponential lower bound on the number of spanning trees of a Steiner triple system.

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1. Introduction

1.1. Spanning trees of 3-uniform hypergraphs

In this paper we investigate the problem of the existence, finding and counting of spanning trees of 3-uniform hypergraphs (henceforth called *3-graphs* for short). The initial motivation for our work was Masbaum and Vaintrob's Pfaffian matrix-tree theorem [21]. They introduce the notion of an orientation (or equivalently a sign) of a spanning tree of a 3-graph. The Pfaffian matrix-tree theorem gives a generating function for signed spanning trees of a 3-graph. We shall be particularly interested in how this spanning tree orientation can be used to identify a large class of 3-graphs for which the problem of counting the number of spanning trees can be done in polynomial time. This class is comparable to that of Pfaffian graphs, for which there is a polynomial-time algorithm for counting the number of perfect matchings. A classical theorem of Kasteleyn [14] is that planar graphs are Pfaffian: can we find a similar class of 3-graphs for which counting the number of spanning trees can be done in polynomial time?

We should be clear at the outset about how we are defining a spanning tree of a 3-graph, for there are various natural alternatives. (More detailed definitions of these and other terms from the theory of hypergraphs are given in Section 2 below.) A spanning tree of a 3-graph H is an inclusion-maximal subset T of the hyperedges of H that covers all the vertices subject to the condition that T does not contain a cycle of hyperedges. If B_H is the usual bipartite vertex-hyperedge incidence graph associated with H, then a spanning tree of H in this sense corresponds precisely to a spanning tree of B_H with the property that either all three edges of B_H incident with a given hyperedge belong to the tree or none of them do. Alternatively, if each hyperedge $\{a, b, c\}$ of H is represented as a triangle of edges ab, bc, ca in a graph G_H on the same vertex set as H, then a spanning tree of H corresponds to a cactus subgraph of G_H covering all vertices. See [1] for a generalization of the Masbaum–Vaintrob theorem to arbitrary hypergraphs in which spanning trees are now cacti with cycles of any odd length and not just triangles.

Spanning trees of 3-graphs differ in fundamental ways from spanning trees of ordinary graphs: a closer correspondence is to be found with perfect matchings, as will become clearer later in the paper. Whereas for spanning trees of graphs the problems of the existence, finding and counting of spanning trees each have a straightforward polynomial-time algorithm, the same is not true for spanning trees of 3-graphs.

As for the algorithmic complexity of matching problems, recall that the augmenting path algorithm finds a maximum matching of a bipartite graph in polynomial time. Consequently, both the problem of whether there is a perfect matching of a bipartite graph and the problem of finding one can be solved in polynomial time. Edmonds' maximum matching algorithm [6] solves in polynomial time the existence and search problems for whether an arbitrary graph has a perfect matching.

Lovász's matroid matching algorithm [16,18] provides a polynomial-time algorithm solving the problem of the existence and finding of a spanning tree of a 3-graph. However, since it solves such a general and complicated problem, the algorithm is involved, has running time a polynomial of high degree and is not optimal when restricting attention from linear matroids to the graphic matroids underlying the case of 3-graphs. The augmenting path algorithm for linear matroids of Gabow and Stallmann [8] has running time $O(mn^2)$ with $O(mn)$ space for graphic matroids of rank n and size m, improved to using $O(m)$ space (alternatively $O(mn\log^6 n)$ time using $O(m\log^4 n)$ space) by the same authors in [7]. In this paper we give some straightforward necessary or sufficient conditions that give simple criteria for the existence of a spanning tree of a 3-graph and in the case of Steiner triple systems a superexponential lower bound on the number of spanning trees.

Our focus then turns to the problem of counting spanning trees of 3-graphs. This problem is #P-complete even for a very restricted class of 3-graphs, which is a consequence of the fact that counting perfect matchings is #P-complete for general graphs [28]. Masbaum and Vaintrob define an orientation or sign of a spanning tree of a 3-graph using orientations of hyperedges in a way that closely follows the definition of the sign of a perfect matching, as elucidated by Hirschman and Reiner [12]. Just as the existence of a Pfaffian orientation of the edges of a graph enables the number of perfect matchings of a graph to be computed in polynomial time, so the existence of what we shall