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# A new basis for osculatory interpolation problems and applications

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ABSTRACT

In this paper we present a polynomial basis based on two-point osculatory interpolation. By exploring some interesting properties of this basis, we derive the smoothness conditions. These conditions can be used for the construction of smooth splines with a low polynomial degree in terms of data points. As an application we give an efficient method for constructing composite splines with shape parameters.



## 1. Introduction

Describing curves is a very important task in mathematics and computer graphics. Generally, the question of representation is of primary importance. Many problems can be solved and many difficulties can be removed by an appropriate choice of the basis. For many practical applications where spline interpolation problems must be solved, it is desirable to obtain such interpolation functions with a low computational cost. It is well known that canonical polynomial basis functions, Bernstein basis and B-Spline models are powerful tools for constructing free-form curves. However, it is not always easy to express the Hermite interpolation in terms of B-spline basis functions (see [16]). Bézier bases are often used to represent Hermite interpolation of small degree (see [5]). Unfortunately, the use of high-degree Bézier curves is not advisable from a calculation point of view. One can use the canonical polynomial basis functions to solve such interpolation problem but the use of this basis requires solving systems with linear equations. Two-point Hermite interpolation method is the most adapted one for solving such problem. Unfortunately, this method has not been fully explored. The term two-point Hermite interpolation is used in the literature to identify the approximations of a function $f$ by a polynomial $p$ in which the values of $f(x)$ and a certain fixed number of its derivatives are fitted by the function values and derivatives of $p(x)$ at two distinct points. In the case where the derivatives are not equidistributed, this method is called Two-point osculatory interpolation.

This paper intends to investigate some important properties of a polynomial basis based on two-point osculatory interpolation in the closed interval [0, 1]. By deriving these polynomial functions we will give the so called the connection coefficients. Using these coefficients we show how to derive smoothness conditions for piecewise polynomial functions in term of the osculatory data breakpoints. Typical applications that could benefit from this construction are osculatory interpolation by spline functions [1,5,11–13], smoothing functions, compressing data [9] and differential equations [7].

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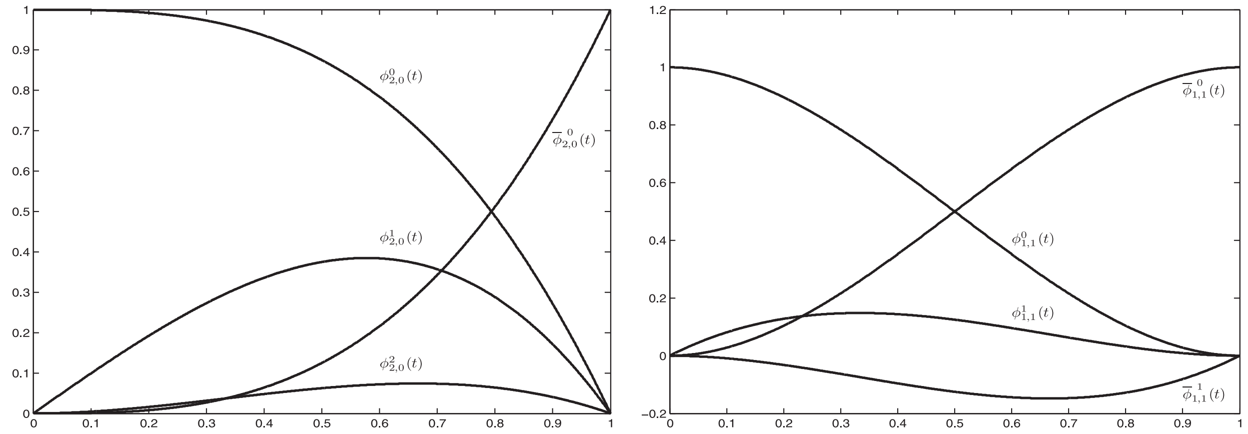


**Fig. 1.** Left: Hermite basis for $r = 2,\ s = 0$. Right: Hermite basis for $r = 1,\ s = 1$.

The sequel of this work is organized as follows. In Sections 2, we introduce the idea of a polynomial basis based on two-point Hermite interpolation and we give its important properties on the interval [0, 1]. In Sections 3, we extend the previous definition to the interval $[a, b]$ and we study the osculatory piecewise Hermite interpolation spline. In Sections 4, we provide a formulation of smoothness conditions for splines in terms of their interpolation data. We also give an application of these results. The last section is devoted to the construction of composite splines with shape parameters.

## 2. Two-point osculatory interpolation polynomial basis

Let $r$ and $s$ be positive integers, for $k = 0, \dots, r$ we define the polynomial functions $\phi^k_{(r,s)}(t)$ in the closed unit interval [0, 1] by

$$\phi^k_{(r,s)}(t) = \frac{(1-t)^{s+1}}{k!} \sum_{j=0}^{r-k} \binom{j+s}{s} t^{j+k}.$$

From this polynomial we define other polynomial functions $\overline{\phi}^k_{(r,s)}(t)$ on [0, 1] by

$$\overline{\phi}^k_{(r,s)}(t) = (-1)^k \phi^k_{(s,r)}(1-t).$$

It is clear that $\phi^k_{(r,s)}(t)$ are positive polynomials of degree $r+s+1$ (Fig. 1). Moreover, if we denote by $\delta_{i,j}$ the Kronecker symbol, then they also satisfy the following properties.

**Proposition 1.** *The polynomial functions $\phi^k_{(r,s)}(t)$ and $\overline{\phi}^k_{(r,s)}(t)$ satisfy the following properties.*

(i) $\frac{d^l}{dt^l}\ \phi^k_{(r,s)}(0) = \delta_{k,l}$ *for $k \le r$ and $0 \le l \le r$.*

(ii) $\frac{d^l}{dt^l}\ \overline{\phi}^k_{(r,s)}(1) = \delta_{k,l}$ *for $k \le s$ and $0 \le l \le s$.*

(iii) $\frac{d^l}{dt^l}\ \phi^k_{(r,s)}(0) = (-1)^{l-k}\frac{l!}{k!}\sum_{j=0}^{r-k}(-1)^j\binom{j+s}{s}\binom{s+1}{l-j-k}$ *for $k \le r$ and $r < l \le r+s+1$.*

(iv) $\frac{d^l}{dt^l}\ \overline{\phi}^k_{(r,s)}(1) = \frac{l!}{k!}\ \sum_{j=0}^{s-k}(-1)^j\binom{j+r}{r}\binom{r+1}{l-j-k}$ *for $k \le s$ and $s \le l \le r+s+1$.*

**Proof.**

(i) Setting $g_n(t) = t^n$, it is easy to see that $\frac{d^l}{dt^l}\ g_n(0) = l!\ \delta_{n,l}$ for any integers $n$ and $l$. Now we consider the auxiliary function $\varphi_{(n,p)}(t) = (1-t)^n\ t^p$, let us prove that

$$\frac{d^l}{dt^l}\varphi_{(n,p)}(0) = \binom{n}{l-p}(-1)^{l-p}\ l!, \quad \text{for any integer} \quad l \le n+p. \tag{1}$$