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Fractal properties of Bessel functions

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ABSTRACT

A fractal oscillatority of solutions of second-order differential equations near infinity is measured by oscillatory and phase dimensions. The phase dimension is defined as a box dimension of the trajectory $(x, \dot{x})$ in $\mathbb{R}^2$ of a solution $x = x(t)$, assuming that $(x, \dot{x})$ is a spiral converging to the origin. In this work, we study the phase dimension of the class of second-order nonautonomous differential equations with oscillatory solutions including the Bessel equation. We prove that the phase dimension of Bessel functions is equal to 4/3, for each order of the Bessel function. A trajectory is a wavy spiral, exhibiting an interesting oscillatory behavior. The phase dimension of a generalization of the Bessel equation has been also computed.

1. Introduction and motivation

Usage of fractal dimension theory in dynamics has evolved into an independent field of mathematics. Main goal is measuring complexity of invariant sets and measures, using fractal dimensions. Fractal dimensions provide a better insight into the dynamics that appear in various problems in physics, meteorology, geology, chemistry, medicine, ecology, and also in engineering, economics, computer science and image processing, as well as in many other branches of mathematics.

Over the years, in mathematics, fractal dimension has been successfully used in studying, for instance, Smale horseshoe, logistic map, Julia and Mandelbrot sets, Lorenz and Hénon attractors, infinite-dimensional dynamical systems, in probability theory, and spiral trajectories, see [1].

Fractal dimensions important for dynamics include, among other dimensions, the Hausdorff dimension and the box dimension. Unlike the Hausdorff dimension, the box dimension is better suited for distinguishing between non-rectifiable smooth curves. Box dimension of a planar curve lying in a neighborhood of some point measures the “amount” of accumulation of the curve in the neighborhood of that point. Generally, the box dimension of this curve is a real number from the interval [1,2]. The other widely used fractal dimension, the Hausdorff dimension, does not distinguish smooth non-rectifiable curves. Hausdorff dimension is countably stable. Taking into account that each smooth non-rectifiable curve is a countable union of rectifiable curves, and the Hausdorff dimension of a rectifiable curve is equal to 1, we see that the Hausdorff dimension of every non-rectifiable curve is equal to 1.

Recently, the fractal oscillatority of solutions of different types of second-order linear differential equations has been considered by Kwong, Pašić, Tanaka and Wong. The Euler type equation has been studied in [2,3], the Hartman–Wintner type equations in [4], half linear equations in [5], and finally the Bessel equation in [6]. In this work, the fractal oscillatority

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is considered in the sense of the oscillatory dimension, at first introduced in Pašić et al., [7]. The oscillatory dimension of a solution $x(t)$ is defined as the box dimension of a graph of function $X(\tau) := x(1/\tau)$ near $\tau = 0$.

On the other hand, the fractal properties of spiral trajectories of dynamical systems in the phase plane have been studied by Žubrinić and Županović, see e.g. [8,9]. From their work the concept of the phase dimension has arisen and has finally been introduced in [7]. They adapted standard idea of phase plane analysis to fractal analysis of solutions of second-order nonlinear autonomous differential equations. Their results show the connection between phase dimension of trajectories near singular points and limit cycles with multiplicity, demonstrated on the standard model for Hopf–Takens bifurcation, see [8]. Similarly, the connection between the asymptotics of the Poincaré map of the planar autonomous system with strictly imaginary eigenvalues, and the phase dimension of the corresponding trajectories, have been shown, see [9].

All these results motivated us to study a fractal connection between the oscillatory and phase dimensions for a class of oscillatory functions, see [10]. In that study, we discovered a specific type of spirals with a nondecreasing radius function, related to chirp-like solutions of a class of equations considered by Kwong, Pašić, Tanaka and Wong, which we call the wavy spirals.

In this article we proved that the phase dimension of Bessel functions is equal to 4/3, for each order of the Bessel function. The phase dimension of a generalization of the Bessel equation has been also computed, depending only on the parameter introduced in generalization.

A model for chirp-like behavior of solutions developed in [10] could not handle the specific behavior of Bessel functions, so to determine the phase dimension of Bessel functions, in this article we generalize the technique used in [10]. Notice that the oscillatory dimension of Bessel functions was already considered in [6]. Here we actually consider some generalization of the Bessel equation that is motivated by the generalization introduced in [6], for whose solutions we determine the phase dimension as our main result, Theorem 1. In order to prove Theorem 1, we first obtain a new version of some theorems from [8]. It is interesting how the phase dimension of solutions depends upon a parameter of this generalization of the Bessel equation. The standard Bessel equation, having phase dimension of Bessel functions equal to 4/3, now becomes a special case. We have done this work hoping that it could help for better understanding of the oscillatority nature of Bessel functions.

For future work we consider to be interesting to examine the connection between the phase dimension of Bessel functions, and the number of limit cycles that can be acquired by small perturbation of the Bessel equation. In relation to results from [8], see Remark 1, in the case of the standard Bessel equation, we expect this number of limit cycles might be 1.

2. Definitions and notation

We first introduce some definitions and notation. For $A \subset \mathbb{R}^N$ bounded we define the *ε-neighborhood* of A by $A_\varepsilon := \{y \in \mathbb{R}^N : d(y, A) < \varepsilon\}$. By the *lower s-dimensional Minkowski content* of A, $s \geq 0$ we mean

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}},$$

and analogously for the *upper s-dimensional Minkowski content* $\mathcal{M}^{*s}(A)$. Now we can introduce the lower and upper box dimensions of A by

$$\underline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}_*^s(A) = 0\}$$

and analogously $\overline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}$. If these two values coincide, we call it simply the box dimension of A, and denote by $\dim_B A$.

For more details on these definitions see e.g. Falconer [8,11,12].

Assume now that x is of class C^1 and $t_0 > 0$. We say that x is a *phase oscillatory* function if the following condition holds: the set $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ in the plane is a spiral converging to the origin.

By the *spiral* here we mean the graph of a function $r = f(\varphi)$, $\varphi \geq \varphi_1 > 0$, in polar coordinates, where

$$\begin{cases} f : [\varphi_1, \infty) \to (0, \infty) \text{ is such that } f(\varphi) \to 0 \text{ as } \varphi \to \infty, \\ f \text{ is } \textit{radially decreasing} \text{ (i.e., for any fixed } \varphi \geq \varphi_1 \\ \text{the function } \mathbb{N} \ni k \mapsto f(\varphi + 2k\pi) \text{ is decreasing).} \end{cases} \tag{1}$$

Depending on the context, by the spiral here we also mean the graph of a function $r = g(\varphi)$, $\varphi \leq \varphi_1' < 0$, in polar coordinates, where for $h(\varphi) = g(-\varphi)$, $\forall \varphi \geq |\varphi_1'|$, the graph of a function $r = h(\varphi)$, $\varphi \geq |\varphi_1'| > 0$, given in polar coordinates, satisfies (1). It is easy to see that the spiral given by function g is a mirror image of the spiral given by function h, regarding x-axis. We also say that a graph of function $r = f(\varphi)$, $\varphi \geq \varphi_1 > 0$, in polar coordinates, is a *spiral near the origin* if there exists $\varphi_2 \geq \varphi_1$ such that a graph of function $r = f(\varphi)$, $\varphi \geq \varphi_2$ is a spiral.

The *phase dimension* $\dim_{ph}(x)$ of function $x : [t_0, \infty) \to \mathbb{R}$ of class C^1 is defined as the box dimension of the corresponding planar curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$.

We use a result for the box dimension of spiral Γ defined by $r = \varphi^{-\alpha}$, $\varphi \geq \varphi_0 > 0$, $\dim_B \Gamma = 2/(1+\alpha)$ when $0 < \alpha \leq 1$, see Tricot [13, p. 121] and some generalizations from [8].

The phase dimension is a fractal dimension, introduced in the study of chirp-like solutions of second order ODEs, see [7]. Fractal dimensions are a well known tool in study of dynamics, see [1].