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A sixth-order family of three-point modified Newton-like multiple-root finders and the dynamics behind their extraneous fixed points

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ABSTRACT

A class of three-point sixth-order multiple-root finders and the dynamics behind their extraneous fixed points are investigated by extending modified Newton-like methods with the introduction of the multivariate weight functions in the intermediate steps. The multivariate weight functions dependent on function-to-function ratios play a key role in constructing higher-order iterative methods. Extensive investigation of extraneous fixed points of the proposed iterative methods is carried out for the study of the dynamics associated with corresponding basins of attraction. Numerical experiments applied to a number of test equations strongly support the underlying theory pursued in this paper. Relevant dynamics of the proposed methods is well presented with a variety of illustrative basins of attraction applied to various test polynomials.

1. Introduction

Newton's method locates a numerical root of a nonlinear equation without difficulty under normal circumstances, provided that a proper initial guess is selected close to the true solution. Unfortunately, it has only linear convergence when locating repeated roots. For repeated roots of a nonlinear equation of the form $f(x) = 0$, given the multiplicity $m \geq 1$ a priori, modified Newton's method [36,37] in the following form

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots \tag{1.1}$$

efficiently locates the desired multiple-root with quadratic-order convergence. It is known that numerical scheme (1.1) is a second-order one-point optimal [23] method on the basis of Kung-Traub's conjecture [23] that any multipoint method [35] without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations. We can find other higher-order multiple-zero finders in a number of literatures [16–18,21,24,25,31,32,40,45].

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Assuming a known multiplicity of $m \geq 1$, we propose in this paper a family of new three-point sixth-order multiple-root finders of modified Newton type in the form of:

$$\begin{cases} y_n = x_n - m \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot Q_f(x_n) \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - m \cdot K_f(x_n) \cdot \dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{1.2}$$

where the desired forms of weight functions Q_f and K_f will be extensively studied for sixth-order of convergence in Section 3. As a consequence, one can regard the last equation in (1.2) as a family of modified Newton-like methods.

The remaining portion of this paper is organized as follows. Section 2 shortly surveys existing studies on multiple-root finders. Fully described in Section 3 is methodology and convergence analysis for newly proposed multiple-root finders. A main theorem on the properties of the family of proposed methods (1.2) is drawn to discover convergence order of six as well as to induce asymptotic error constants and error equations by use of a family of weight functions Q_f and K_f dependent on two principal roots of function-to-function ratios. In Section 4, special cases of weight functions are considered based on polynomials and low-order rational functions. Section 5 extensively investigates the extraneous fixed points and related dynamics underlying the basins of attraction. Tabulated in Section 6 are computational results for a variety of numerical examples. Table 5 compares the magnitudes of $e_n = x_n - \alpha$ of the proposed methods with those of a member of an existing sixth-order family of methods. Dynamical characteristics of the proposed methods along with their illustrative basins of attraction are depicted at great length with detailed analyses, comparisons and comments. Briefly stated at the end is overall conclusion together with a possible development of future work.

2. Review of existing sixth-order multiple-root finders

The orders of convergence of existing multiple-root finders are mostly found to be less than or equal to 4, and more higher-order multiple-root finders are rarely to be found. Very recently Geum–Kim–Neta [19] have developed a class of two-point sixth-order multiple-root finders by extending the classical modified double-Newton method with extensive analysis of their relevant dynamics behind the basins of attraction from the viewpoint of the extraneous fixed points. One member of the class is introduced as follows shown by (2.1):

Let a function $f : \mathbb{C} \to \mathbb{C}$ have a repeated zero α with integer multiplicity $m > 1$ and be analytic [1] in a small neighborhood of α.

$$\begin{cases} y_n = x_n - m\dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \dfrac{m + a_1 u}{1 + b_1 u + b_2 u^2} \times \dfrac{1}{1 + 2(m-1)t} \cdot \dfrac{f(y_n)}{f'(y_n)}, u = \left[\dfrac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, t = \left[\dfrac{f'(y_n)}{f'(x_n)}\right]^{\frac{1}{m-1}}, \end{cases} \tag{2.1}$$

where $a_1 = \frac{2m(4m^4-16m^3+31m^2-30m+13)}{(m-1)(4m^2-8m+7)}$, $b_1 = \frac{4(2m^2-4m+3)}{(m-1)(4m^2-8m+7)}$ and $b_2 = -\frac{4m^2-8m+3}{4m^2-8m+7}$. This member will be compared with another family of sixth-order multiple-root finders to be developed in the next section of this paper.

3. Methodology and convergence analysis

We assume that a function $f : \mathbb{C} \to \mathbb{C}$ has a repeated zero α with integer multiplicity $m \geq 1$ and is analytic in a small neighborhood of α. Given an initial guess x_0 sufficiently close to α, new three-point iterative methods proposed in (1.2) to find an approximate zero α of multiplicity m will take the specific form of:

$$\begin{cases} y_n = x_n - m \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ w_n = x_n - m \cdot Q_f(s) \cdot \dfrac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = x_n - m \cdot K_f(s, v) \cdot \dfrac{f(x_n)}{f'(x_n)}, \end{cases} \tag{3.1}$$

where

$$s = \left[\frac{f(y_n)}{f(x_n)}\right]^{\frac{1}{m}}, \tag{3.2}$$

$$v = \left[\frac{f(w_n)}{f(x_n)}\right]^{\frac{1}{m}}, \tag{3.3}$$