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Asymptotic expansions related to hyperfactorial function and Glaisher–Kinkelin constant

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A B S T R A C T

In this paper, by the Bernoulli numbers and the exponential complete Bell polynomials, we establish two general asymptotic expansions related to the hyperfactorial function and the Glaisher–Kinkelin constant, where the coefficients in the series of the expansions can be determined by recurrences. Moreover, the explicit expressions of the coefficients are studied and some special asymptotic expansions are presented.

1. Introduction

The Glaisher–Kinkelin constant $A = 1.2824271291\ldots$ is defined by the limit

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \frac{H(n)}{n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}}}, \tag{1.1}$$

where $H(n) = \prod_{k=1}^{n} k^k$ is the hyperfactorial function. This constant can also be defined by

$$A = \lim_{n\to\infty} \frac{(2\pi)^{\frac{n}{2}} n^{\frac{n^2}{2}-\frac{1}{12}} e^{-\frac{3n^2}{4}+\frac{1}{12}}}{G(n+1)},$$

where $G(z)$ is the Barnes G-function and satisfies the equation $G(z+1) = \Gamma(z)G(z)$ with $G(1) = 1$, and $\Gamma(z)$ is the familiar gamma function.

Besides the hyperfactorial function and the Barnes G-function, the Glaisher–Kinkelin constant A is closely related to some other special functions and mathematical constants such as the Riemann zeta function $\zeta(s)$ and the Euler–Mascheroni constant γ. For example, the following closed-form representations of A hold true:

$$A = \exp\left(\frac{1}{12} - \zeta'(-1)\right) = \exp\left(-\frac{\zeta'(2)}{2\pi^2} + \frac{\ln(2\pi)+\gamma}{12}\right).$$

For more properties on the Glaisher–Kinkelin constant and the Barnes G-function, the readers are referred to, for example, [2,12,15–17,33].

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Recently, by the Euler–Maclaurin formula, Chen [6] presented the following asymptotic expansion for the hyperfactorial function and the Glaisher–Kinkelin constant:

$$\begin{aligned}\ln A_n &= \sum_{k=1}^{n} k\ln k - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}\\ &\sim \ln A - \sum_{k=1}^{\infty}\frac{B_{2k+2}}{2k(2k+1)(2k+2)}\frac{1}{n^{2k}},\end{aligned}$$

as $n\to\infty$, where B_n are the Bernoulli numbers. This expansion can be rewritten as

$$1^1 2^2 \ldots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \exp\left\{\sum_{k=1}^{\infty}\frac{-B_{k+2}}{k(k+1)(k+2)}\frac{1}{n^k}\right\}, \quad n\to\infty, \tag{1.2}$$

because $B_{2k+1}=0$ for $k=1,2,\ldots$. Substituting the values of B_n in (1.2) yields

$$\begin{aligned}&1^1 2^2 \ldots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}}\\ &\times \exp\left\{\frac{1}{720n^2}-\frac{1}{5040n^4}+\frac{1}{10080n^6}-\frac{1}{9504n^8}+\frac{691}{3603600n^{10}}-\frac{1}{1872n^{12}}+\cdots\right\}, \quad n\to\infty.\end{aligned}$$

Mortici [26] also established (1.2) and gave a recurrence relation to compute the coefficients of the series in the formula. Using (1.2), Chen and Lin [8] obtained a general asymptotic expansion:

$$1^1 2^2 \ldots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty}\frac{\breve{\alpha}_k}{n^k}\right)^{\frac{1}{r}}, \quad n\to\infty, \tag{1.3}$$

and presented the expression for the coefficient sequence $(\breve{\alpha}_k)$. Moreover, by considering the asymptotic expansions of the Barnes G-function, Chen [4] further gave another general expansion:

$$1^1 2^2 \ldots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty}\frac{\breve{\varphi}_k}{n^k}\right)^{\frac{n^l}{r}}, \quad n\to\infty, \tag{1.4}$$

where l is a positive integer. For more results on asymptotic expansions and inequalities of the hyperfactorial function, the Glaisher–Kinkelin constant, and the related Barnes G-function and Bendersky constants, the readers may consult, for example, the papers due to Cheng and Chen [10], Choi [11], Ferreira and López [14], Lin [19], Lu and Mortici [20], and Mortici [24].

Inspired by these works, in this paper, we present the following two general asymptotic expansions for the hyperfactorial function and the Glaisher–Kinkelin constant:

$$1^1 2^2 \ldots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty}\frac{\alpha_k}{(n+h)^k}\right)^{\frac{1}{r}}, \tag{1.5}$$

$$1^1 2^2 \ldots n^n \sim A\cdot n^{\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}} e^{-\frac{n^2}{4}} \left(\sum_{k=0}^{\infty}\frac{\varphi_k}{(n+h)^k}\right)^{\frac{n}{r}+q}, \tag{1.6}$$

as $n\to\infty$. This paper is organized as follows. In Section 2, we present the first general asymptotic expansion (1.5) by the exponential complete Bell polynomials, improve Chen and Lin's result (1.3), and give some special asymptotic expansions by specifying the parameters. In Section 3, we establish the explicit expression of the coefficient sequence in the series of (1.5) by the generating function method and the properties of Bernoulli numbers and Bell polynomials. Finally, in Section 4, we present the second general asymptotic expansion (1.6) and further discuss its special cases. It can be found that the Glaisher–Kinkelin constant A and the hyperfactorial function $H(n)$ play the same roles in (1.1) as the constant $\sqrt{2\pi}$ and the factorial function play in the Stirling formula

$$\sqrt{2\pi} = \lim_{n\to\infty}\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}.$$

Therefore, we also list in this paper some similar asymptotic expansions of the gamma function and the factorial function so that the readers can compare them conveniently.

2. The first general asymptotic expansion

The exponential complete Bell polynomials Y_n are defined by

$$\exp\left(\sum_{m=1}^{\infty}x_m\frac{t^m}{m!}\right) = \sum_{n=0}^{\infty}Y_n(x_1,x_2,\ldots,x_n)\frac{t^n}{n!} \tag{2.1}$$