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Generalized viscosity approximation methods for mixed equilibrium problems and fixed point problems

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ABSTRACT

In this paper, we present a new iterative method based on the hybrid viscosity approximation method and the hybrid steepest-descent method for finding a common element of the set of solutions of generalized mixed equilibrium problems and the set of common fixed points of a finite family of nonexpansive mappings in Hilbert spaces. Furthermore, we prove that the proposed iterative method has strong convergence under some mild conditions imposed on algorithm parameters. The results presented in this paper improve and extend the corresponding results reported by some authors recently.

1. Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let C be a nonempty closed convex subset of H.

Recently, Peng and Yao [12] considered the following generalized mixed equilibrium problem, which involves finding $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where A: $H \to H$ is a nonlinear mapping, $\varphi : C \to \mathbb{R}$ is a function and $\Theta : C \times C \to \mathbb{R}$ is a bifunction. The solution set of (1.1) is denoted by Ω.

If $A = 0$, then problem (1.1) becomes the following mixed equilibrium problem: Finding $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0, \quad \forall y \in C,$$

which was studied by Ceng and Yao [5].

If $\varphi = 0$, then problem (1.1) reduces to the following generalized equilibrium problem: Finding $x^* \in C$ such that

$$\Theta(x^*, y) + \langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C,$$

which was introduced and studied by Takahashi and Takahashi [15].

If $\Theta = \varphi = 0$, then problem (1.1) collapses to the classical variational inequality, which involves finding $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

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The theory of variational inequality emerged as a rapidly growing area of research because of its applications in nonlinear analysis, optimization, economics, game theory (see [1–4] and the references cited therein).

Let T: $C \to C$ be a nonlinear mapping. We use Fix(T) to denote the set of fixed points of T, *i.e.*, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. A mapping is called nonexpansive if the following inequality holds:

$$\|Tx - Ty\| \leq \|x - y\|$$

for all x, $y \in C$.

In 1967, Halpern [9] considered the following explicit iterative process:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad \forall n \geq 0,$$

where u is a given point and T: $C \to C$ is nonexpansive. He proved the strong convergence of $\{x_n\}$ to a fixed point of T provided that $\alpha_n = n^{-\theta}$ with $\theta \in (0, 1)$. In 2003, Xu [17] introduced the following iterative process:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n A) T x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in (0, 1). He proved the above sequence $\{x_n\}$ converges strongly to the unique solution of the minimization problem with $C = \mathrm{Fix}(T)$: $\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle$, where A is a strongly positive bounded linear operator on H.

In 2006, Marino and Xu [11] considered the following viscosity iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T x_n, \quad \forall n \geq 0,$$

where f is a contraction on H. They proved the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

In 2001, Yamada et al. [19] considered the following hybrid steepest-descent iterative method:

$$x_{n+1} = Tx_n - \mu \lambda_n F(Tx_n),$$

where F is κ-Lipschitzian continuous and η-strongly monotone operator with $\kappa > 0$, $\eta > 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$. Under some appropriate conditions, the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

In [16], Tian considered the following general viscosity type iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F) T x_n, \quad \forall n \geq 0.$$

Under certain approximate conditions, the above sequence $\{x_n\}$ converges strongly to a fixed point of T, which solves the variational inequality

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T).$$

In 2014, Zhou and Wang [21] proposed a simple explicit iterative algorithm for finding a solution of variational inequality over the set of common fixed points of a finite family nonexpansive mappings. They introduced an explicit scheme as follows:

Theorem 1.1. *Let H be a real Hilbert space and F: $H \to H$ be an κ-Lipschitzian continuous and η-strongly monotone mapping with $\kappa > 0$ and $\eta > 0$. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $C = \cap_{i=1}^N \mathrm{Fix}(T_i) \neq \phi$. For any point $x_0 \in H$, define a sequence $\{x_n\}$ as follows:*

$$x_{n+1} = (1 - \lambda_n \mu F) T_N^n T_{N-1}^n, \ldots, T_1^n x_n, \quad \forall n \geq 0, \tag{1.3}$$

where $\mu \in (0, \frac{2\eta}{\kappa^2})$ and $T_i^n = (1 - \sigma_n^i)I + \sigma_n^i T_i$ for $i = 1, 2, \ldots, N$. When the parameters satisfy appropriate conditions, the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality (1.2).

Recently, Zhang and Yang [20] proposed an explicit iterative algorithm based on the viscosity method for finding a solution for a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings as follows:

Theorem 1.2. *Let H be a real Hilbert space and F: $H \to H$ be an κ-Lipschitzian continuous and η-strongly monotone mapping with $\kappa > 0$ and $\eta > 0$. Let $\{T_i\}_{i=1}^N$ be N nonexpansive mappings of H such that $C = \cap_{i=1}^N \mathrm{Fix}(T_i) \neq \phi$ and V be an ρ-Lipschitzian on H with $\rho > 0$. For any point $x_0 \in H$, define a sequence $\{x_n\}$ in the following manner:*

$$x_{n+1} = \alpha_n \gamma V(x_n) + (I - \alpha_n \mu F) T_N^n T_{N-1}^n, \ldots, T_1^n x_n, \quad \forall n \geq 0,$$

where $0 < \gamma\rho < \tau$ with $\tau = \mu(2\eta - \mu\kappa^2)$, $0 < \mu < \frac{2\eta}{\kappa^2}$, $T_i^n = (1 - \sigma_n^i)I + \sigma_n^i T_i$ for $i = 1, 2, \ldots, N$ and $\sigma_n^i \in (\zeta_1, \zeta_2)$ for some ζ_1, $\zeta_2 \in (0, 1)$. When the parameters satisfy appropriate conditions, then the sequence $\{x_n\}$ converges strongly to the unique solution $x^* \in C$ of the variational inequality:

$$\langle (\mu F - \gamma V)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \cap_{i=1}^N \mathrm{Fix}(T_i).$$

In this paper, motivated by the above works, we introduce a new iterative algorithm like viscosity approximation and investigate fixed points of nonexpansive mappings and solutions of equilibrium problem (1.1). Strong convergence theorems for common elements are established in Hilbert spaces.