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Operational matrix approach for the solution of partial integro-differential equation

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ABSTRACT

In this paper, an effective numerical method is introduced for the treatment of Volterra singular partial integro-differential equations. They are based on the operational and almost operational matrix of integration and differentiation of 2*D* shifted Legendre polynomials. The methods convert the singular partial integro-differential equation in to a system of algebraic equations. Convergence analysis and error estimates are derived for the proposed method. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

Volterra type integral equations arise in a variety of science and technology fields [1]. An area of increasing scientific interest over the past decades is the study of Volterra integro-differential equation. The theory and application of Volterra integro-differential equations play an important role in the mathematical modeling of many fields: physical phenomena, biological model, chemical kinetics, engineering science and application in heat flow [12]. Several numerical methods for approximating the solution of Volterra integral equations with weakly singular kernel are known [2–10,13]. Also various numerical techniques have been developed for the solution of partial integro-differential equations, see for examples, [11,14,15,17–19].

The main aim of this paper is to study second kind Volterra singular partial integro-differential equation of the form

$$\begin{aligned} &u_y = u(x,y) + f(x,y) + \int_0^x \int_0^y \frac{G(u(s,t))}{(x-s)^\alpha} dt ds \quad 0 \le s \le x, \quad 0 \le t \le y \\ &(x,y) \in [0,1] \times [0,1] \\ &0 < \alpha < 1 \end{aligned} \tag{1}$$

with initial condition $u(x,0) = u_0(x)$.

Where, u is unknown function in $\Omega(= [0,1] \times [0,1])$ which should be determined and the functions f, $u_0(x)$ are known. G is a linear or non-linear differential operator. It can be seen that in above equation the kernel function has a singularity at the origin. The functions $f(x, y)$ and $u(x, y)$ are assumed to be sufficiently smooth in order to guarantee the existence and uniqueness of a solution $u \in C([0, 1] \times [0, 1])$. Equations of the form (1) are usually difficult to solve analytically so it is required to obtain an efficient approximation or numerical methods.

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In this paper, numerical solution of (1) is computed by using the 2*D* shifted orthogonal Legendre polynomial system. The advantage of orthogonal system, proposed in present work is that the Legendre bivariate polynomial provide an accurate approximation of the problem solution with reduced number of basis functions.

This paper is organized as follows: Section 2 represents preliminaries which is devoted to used polynomial and function approximation. In this section we introduced 2*D* shifted Legendre polynomials and some properties of them. In Section 3 we constructed operational matrices of differentiation and almost operational matrices of integration. In Section 4 we discussed two kinds of problems for the cases $Gu = u_{yy}$ and $Gu = u_{xy}$ and the proposed method is used to approximate the singular partial integro-differential equation. In Sections 5 and 6 we derived the convergence analysis and error estimates for our proposed method. In Section 7 we demonstrate the accuracy of the proposed method by considering several test examples. Finally, some concluding remarks are given in Section 8.

2. Preliminaries: Used polynomials and function approximation

2.1. Definition and properties of 2D shifted Legendre polynomials

The 2*D* shifted Legendre polynomials are defined on $\Omega(= [0, 1] \times [0, 1])$ as follows:

$$\psi_{nm}(x, y) = L_n(2x - 1)L_m(2y - 1), \quad n, m = 0, 1, 2, \ldots, \tag{2}$$

where, $L_n(x)$ and $L_m(x)$ are the well-known Legendre polynomials, respectively of order n and m, which are defined on the interval $[-1, 1]$ and satisfy the following recursive formula

$$L_0(x) = 1, L_1(x) = x \quad \text{and} \quad L_{m+1}(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x), \quad m = 1, 2, 3, \ldots$$

and form a complete basis over the interval $[-1, 1]$. 2*D* shifted Legendre polynomials are orthogonal with respect to weight function ω(x,y) such that

$$\int_0^1 \int_0^1 \omega(x, y)\psi_{nm}(x, y)\psi_{ij}(x, y)dxdy = \begin{cases} \dfrac{1}{(2n+1)(2m+1)}, & \text{for} \quad \imath = n, \jmath = m, \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Suppose $X = L^2(\Omega)$ be the inner product space. Then the inner product in this space is defined by

$$\langle f(x, y), g(x, t)\rangle = \int_0^1 \int_0^1 f(x, y)g(x, y)dxdy$$

and the norm is as follows

$$\| f(x, y) \|_2 = \langle f(x, y), f(x, t)\rangle^{1/2} = \left(\int_0^x \int_0^y |f(x, y)|^2 dxdy\right)^{1/2}$$

2.2. Function approximation

Suppose that $f(x, y)$ is an arbitrary function in $L^2(\Omega)$, then it can be approximated as

$$f(x, y) \approx \sum_{n=0}^{N} \sum_{m=0}^{N} F_{nm}\psi_{nm}(x, y) = F^T \Psi(x, y), \tag{4}$$

where, F and Ψ are $(N+1)^2 \times 1$ vector given by

$$F = [F_{00}, F_{01}, \ldots, F_{0N}, F_{10}, F_{11}, \ldots, 1_{1N}, \ldots, F_{N0}, \ldots, F_{NN}]^T, \tag{5}$$

$$\Psi(x, y) = [\psi_{00}(x, y), \psi_{01}(x, y), \ldots, \psi_{0N}(x, y), \psi_{10}(x, y), \psi_{11}(x, y), \ldots, \psi_{1N}(x, y), \ldots, \psi_{N0}(x, y), \ldots, \psi_{NN}(x, y)]^T \tag{6}$$

and, $\psi_{nm}(x, y) = \psi_n(x)\psi_m(y)$

Theorem. *Let $f_N(x, y) = F_N^T \Psi(x, y)$ be the 2D shifted Legendre expansion of real sufficiently smooth function $f(x, y)$ in Ω, where*

$$F_N = [f_{00}, f_{01}, \ldots, f_{0N}, f_{10}, f_{11}, \ldots, f_{1N}, \ldots, f_{N0}, \ldots, f_{NN}]^T$$

and,

$$f_{nm} = (2n+1)(2m+1) \int_0^1 \int_0^1 f(x, y)\psi_{nm}(x, y)dxdy$$

then there is a real number α such that

$$\| f(x, y) - f_N(x, y) \|_2 \leq \frac{\alpha}{(N+1)!2^{2N+1}}$$

.

Proof. See the reference [7]. □