Applied Mathematics and Computation 283 (2016) 208–215

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Diagonal update method for a quadratic matrix equation

Young-Jin Kim*, Hyun-Min Kim

Department of Mathematics, Pusan National University, Busan, 609-735, Republic of Korea

ARTICLE INFO

MSC:
65H10

Keywords:
Quadratic matrix equation
M-matrix
Fixed-point iteration
Bernoulli's method
Diagonal update method

ABSTRACT

The quadratic matrix equation $AX^2 + BX + C = 0$ has been studied in many researches. The sufficient condition for the existence of the primary solution is already provided where the equation arose from an overdamped vibrating system with *M*-matrix coefficients. The Bernoulli's method can not be modified by relaxation in this equation. In this paper, we will introduce the diagonal update method and modify the Bernoulli's method.

1. Introduction

Quadratic matrix equations (QME) appear in many scientific areas. In this paper, we consider the following quadratic matrix equation

$$Q_1(X) = AX^2 + BX + C = 0 \tag{1.1}$$

where

$A \in \mathbb{R}^{n\times n}$ is a diagonal matrix with positive diagonal elements,
$B \in \mathbb{R}^{n\times n}$ is a nonsingular *M*-matrix,
$C \in \mathbb{R}^{n\times n}$ is an *M*-matrix such that $B^{-1}C \geq 0$.

There are many methods for finding numerical solvent for various QME. Newton's method was employed by Davis [3]. The global convergence of Newton's method was improved by Higham and Kim [10]. He et al. suggested a cyclic reduction algorithm for solving a Quasi-Birth–Death problem [9]. A successive approximation method and a Newton's method for QME were constructed by Bai et al. [2]. They also proved the convergence of those methods. Bai and Gao suggest a modified Bernoulli's method for finding a minimal solvent [1]. In addition, there are many researches for other iterative methods and various solvents of QME [5,7,8,15].

The condition of coefficient matrices of the above QME is from Yu [16]. It is motivated by a quadratic eigenvalue problem arising from an overdamped vibrating system [12,13]. Yu [16] suggested the sufficient condition for the existence of the primary solvent by constructing fixed-point iterations. These iterations were produced by similar way in [4].

This equation can be easily modified as following $Q_2(X) = 0$ by multiplying A^{-1} [16],

$$Q_2(X) = X^2 + BX + C = 0. \tag{1.2}$$

* Corresponding author. Tel.: +82 1051342806.
E-mail addresses: kyj2806@hanmail.net, kyj2806@gmail.com (Y.-J. Kim), hyunmin@pusan.ac.kr (H.-M. Kim).

http://dx.doi.org/10.1016/j.amc.2016.02.016
0096-3003/

This equation also preserves the M-matrix structure of $Q_1(X)$. For the convenience, we will consider $Q_2(X) = 0$ though $Q_1(X)$. We can construct the fixed-point iterative methods as follows:

$$\begin{cases} \text{Given } X_0 \in \mathbb{R}^{n\times n} \\ X_{k+1} = \mathcal{F}_i(X_k), \end{cases} \quad k = 0, 1, 2, \ldots \tag{1.3}$$

where

$$\mathcal{F}_1(X) = -(B+X)^{-1}C, \tag{1.4}$$

$$\mathcal{F}_2(X) = -B^{-1}(X^2 + C). \tag{1.5}$$

Yu [16] also showed the monotone convergence of (1.4) and (1.5). We apply the relaxation skill to (1.4) and (1.5) to derive the following two functions for some $0 \le a \le 1$.

$$\mathcal{F}_1(X) = aX - (1-a)(B+X)^{-1}C \tag{1.6}$$

$$\mathcal{F}_2(X) = aX - (1-a)B^{-1}(X^2 + C) \tag{1.7}$$

Example 1.1. In this example, we will compare the iteration numbers of relaxed functions $\mathcal{F}_1$ and $\mathcal{F}_2$. This example is well-known example and let $n = 30$.

$$A = I, \quad B = \begin{bmatrix} 20 & -10 & & & & \\ -10 & 30 & -10 & & & \\ & -10 & 30 & -10 & & \\ & & -10 & \ddots & \ddots & \\ & & & \ddots & 30 & -10 \\ & & & & -10 & 20 \end{bmatrix},$$

$$C = \begin{bmatrix} 15 & -5 & & & & \\ -5 & 15 & -5 & & & \\ & -5 & 15 & -5 & & \\ & & -5 & \ddots & \ddots & \\ & & & \ddots & 15 & -5 \\ & & & & -5 & 15 \end{bmatrix}.$$

a	0	0.1	0.2	0.3
iter of $\mathcal{F}_1$	15	22	30	43
iter of $\mathcal{F}_2$	18	30	54	Divergence

We can easily verify that the relaxation skill cannot reduce iteration numbers of $\mathcal{F}_1$ and $\mathcal{F}_2$.
So we apply the diagonal update skill and suggest sufficient condition through relaxation skill.

$$\mathcal{F}_1(X) = -(B + X - \delta_X I)^{-1}(C + \delta_X X), \tag{1.8}$$

$$\mathcal{F}_2(X) = -(B - 2\delta_X I)^{-1}(X^2 + 2\delta_X X + C), \tag{1.9}$$

where $\delta_X = \min\{1, \min\{|X_{i,i}|\}\}$.

2. Related definitions and lemmas

In this section, we introduce some well-known definitions [6,11,14] and lemmas. These definitions and lemmas are useful to understand M-matrix structure and the existence of primary solvent of QME (1.2) on the sufficient condition.

Definition 2.1. A matrix $A \in \mathbb{R}^{n\times n}$ is an M-matrix if $A = sI - B$ for some nonnegative B and s with $s \ge \rho(B)$ where ρ is the spectral radius ; it is a singular M-matrix if $s = \rho(B)$ and a nonsingular M-matrix if $s > \rho(B)$.

Lemma 2.2. *For a Z-matrix A, the following are equivalent:*

(1) *A is a nonsingular M-matrix.*
(2) *A^{-1} is nonnegative.*
(3) *$Av > 0$ for some vector $v > 0$.*
(4) *All eigenvalues of A have positive real parts.*