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Smoothing transformation and spline collocation for linear fractional boundary value problems ☆

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ABSTRACT

We construct and justify a high order method for the numerical solution of multi-point boundary value problems for linear multi-term fractional differential equations involving Caputo-type fractional derivatives. Using an integral equation reformulation of the boundary value problem we first regularize the solution by a suitable smoothing transformation. After that we solve the transformed equation by a piecewise polynomial collocation method on a mildly graded or uniform grid. Optimal global convergence estimates are derived and a superconvergence result for a special choice of collocation parameters is established. To illustrate the reliability of the proposed method some numerical results are given.

1. Introduction

Differential equations involving differential operators of fractional (non-integer) order have been proved to be a valuable tool in modeling many phenomena in the fields of physics, chemistry, engineering and others (see, for example, [1–3]). Mathematical aspects of fractional differential equations and methods of their solution were discussed by many authors (see, for example, [4–6] and references cited therein). A lot publications are devoted to the numerical solution of fractional initial value problems (see, e.g., [7–13]). In the last decade also boundary value problems for fractional differential equations have received an increasing attention. In particular, various existence and uniqueness results for fractional boundary value problems are obtained in [14–19] and the numerical solution of boundary value problems for fractional differential equations is considered in [20–30].

In the present paper we study the convergence behavior of a modified spline collocation method for the numerical solution of fractional boundary value problems of the form

$$(D_*^{\alpha_p} y)(t) + \sum_{i=0}^{p-1} d_i(t)(D_*^{\alpha_i} y)(t) = f(t)\,, \quad 0 \le t \le b, \tag{1.1}$$

$$\sum_{j=0}^{n_0} \beta_{ij0}\, y^{(j)}(0) + \sum_{k=1}^{l}\sum_{j=0}^{n_1} \beta_{ijk}\, y^{(j)}(b_k) = \gamma_i\,, \quad i = 0, \ldots, n-1\,, \;\; n := \lceil \alpha_p \rceil, \tag{1.2}$$

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where $\beta_{ij0}, \beta_{ijk}, \gamma_i \in \mathbb{R} := (-\infty, \infty)$, $\lceil \alpha \rceil$ is the smallest integer greater or equal to $\alpha \in \mathbb{R}$,

$$\begin{gathered} 0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_p \le n\,, \quad 0 < b_1 < \cdots < b_l \le b\,, \\ p, l \in \mathbb{N} := \{1, 2, \ldots\}\,, \quad n_0, n_1 \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \quad n_0 < n, \quad n_1 < n, \end{gathered} \tag{1.3}$$

$d_i : [0, b] \to \mathbb{R}$ $(i = 0, \ldots, p-1)$ and $f : [0, b] \to \mathbb{R}$ are some given continuous functions, and $D_*^{\alpha_i} y$ $(i = 0, \ldots, p)$ are Caputo derivatives of an unknown function y. The Caputo fractional differential operator D_*^α of order $\alpha > 0$ is defined by the formula (see, e.g, [4])

$$(D_*^\alpha y)(t) = (D^\alpha (y - \mathcal{Q}_{k-1}[y]))(t), \quad t > 0, \quad k := \lceil \alpha \rceil,$$

where

$$\mathcal{Q}_{k-1}[y](s) = \sum_{i=0}^{k-1} \frac{y^{(i)}(0)}{i!} s^i$$

and D^α is the Riemann–Liouville fractional differentiation operator of order $\alpha > 0$:

$$(D^\alpha y)(t) = \frac{d^k}{dt^k} (J^{k-\alpha} y)(t), \quad t > 0, \quad k := \lceil \alpha \rceil.$$

Here J^α, the Riemann–Liouville integral operator of order $\alpha > 0$, is defined by the formula

$$(J^\alpha y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds, \quad t > 0, \tag{1.4}$$

where Γ is the Euler gamma function. For $\alpha = 0$ we set $D^0 = D_*^0 = J^0 := I$ where I is the identity mapping. If $\alpha = k \in \mathbb{N}$ then $D^k y = D_*^k y = y^{(k)}$ where $y^{(k)}$ is the usual k-th order derivative of y.

It is well known (see, e.g., [31]) that $J^\alpha, \alpha > 0$, is linear, bounded and compact as an operator from $L^\infty(0, b)$ into $C[0, b]$. Moreover (see, e.g., [5]), we have for any $y \in L^\infty(0, b)$ that

$$(J^\alpha y)^{(k)} \in C[0, b], \quad (J^\alpha y)^{(k)}(0) = 0, \quad \alpha > 0, \quad k = 0, \ldots, \lceil \alpha \rceil - 1, \tag{1.5}$$

$$J^\alpha J^\beta y = J^{\alpha+\beta} y, \quad \alpha > 0, \quad \beta > 0, \tag{1.6}$$

$$D^\beta J^\alpha y = D_*^\beta J^\alpha y = J^{\alpha-\beta} y, \quad 0 < \beta \le \alpha. \tag{1.7}$$

As a rule, initial and boundary value problems for fractional differential equations are equivalent to certain weakly singular integral equations of the second kind. More exactly, initial value problems correspond to Volterra type integral equations and boundary value problems to Fredholm type integral equations (see, e.g., [4,7,21]). Therefore in general we cannot expect that a solution of a fractional initial or boundary value problem is smooth on the closed interval of integration [0, b] since the derivatives of solutions of weakly singular integral equations of the second kind are typically unbounded near the boundary of [0, b]. Actually, the solutions of weakly singular Volterra equations are typically non-smooth at the initial point 0 of the interval of integration [0, b] whereas the solutions of weakly singular Fredholm equations are typically non-smooth at both endpoints of [0, b] (see, e.g., [32,33]). Due to the lack of smoothness of the exact solution, piecewise polynomial collocation methods based on uniform grids for solving such equations show slow convergence behavior [32]. In order to construct methods with higher convergence order it is necessary to take into account the possible singular behavior of the exact solution.

From Theorem 2.1 below we see that the derivatives of the solution of Problem (1.1) and (1.2) and corresponding to this problem Fredholm integral equation are unbounded only near the left endpoint 0 of the interval of integration [0, b]. Therefore we can apply for solving Problem (1.1) and (1.2) also some methods which are typical for solving Volterra equations. In particular, we can construct high order collocation methods by using polynomial splines and special graded grids where the grid points are more densely clustered near the singular point $t = 0$ of the exact solution $y(t)$ of Problem (1.1) and (1.2) [27]. However, this approach has a disadvantage since the use of strongly non-uniform grids with special subintervals of very small length near the singular point of the exact solution may cause serious rounding error problems and because of loss of precision may lead to unstable behavior of numerical results. To diminish loss of precision we may resort collocation on the uniform grid using non-polynomial basic functions which reflect the singular behavior of the exact solution. This approach has been applied for solving Volterra integral equations in [34–36] and for solving fractional differential and integro-differential equations in [37–39].

In the present paper we use an alternative approach for diminishing loss of precision. First we introduce an integral equation reformulation of Problem (1.1) and (1.2). Then we perform in the integral equation the smoothing change of variables $t = b^{1-\rho}\tau^\rho$ $(\rho \in [1, \infty))$ so that the singularities of the (usual) derivatives of the exact solution will be milder or disappear. After that we solve the transformed integral equation by a piecewise polynomial collocation method on mildly graded or uniform grid. The final step of our method is based on a conversion of the obtained spline approximations into (typically non-polynomial) approximations for the solution of Problem (1.1) and (1.2). Similar ideas for solving Volterra equations are used in [40–44] and for solving fractional differential and integro-differential equations in [45,46].