Applied Mathematics and Computation 283 (2016) 299–315

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

New approximations of some expressions involving trigonometric functions

Marija Nenezić[a], Branko Malešević[a], Cristinel Mortici[b,c,*]

[a] Faculty of Electrical Engineering, University of Belgrade, Bulevar Kralja Aleksandra 73, 11000 Belgrade, Serbia
[b] Valahia University of Târgovişte, Bd. Unirii 18, 130082 Târgovişte, Romania
[c] Academy of Romanian Scientists, Splaiul Independenţei 54, 050094 Bucharest, Romania

A R T I C L E I N F O

Keywords:
Wilker inequality
Trigonometric approximation

A B S T R A C T

The aim of this paper is to apply a computation method due to Malešević and Makragić (Malešević and Makragić, 2016) for approximating some trigonometric functions. Inequalities of Wilker–Cusa–Huygens are discussed, but the method can be successfully applied to a wide class of problems. In particular, we improve the estimates recently obtained by Mortici (Mortici, 2014) and moreover we show that they hold true also on some extended intervals.

1. Introduction

Wilker [2] presented the inequality

$$2 < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x},$$

for $x \in (0, \pi/2)$ and he asked for the largest constant $c > 0$ in

$$2 + cx^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x},$$

and $x \in (0, \pi/2)$. The Wilker inequality has attracted the interest of several authors in the recent past. In particular Sumner et al. [3] proved the following double inequality

$$2 + \frac{16}{\pi^4}x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^3 \tan x,$$

for $x \in (0, \pi/2)$. Mortici [1] has proved the following two statements:

Theorem 1.1. *For every* $x \in (0, 1)$ *we have*:

$$2 + \left(\frac{8}{45} - a(x)\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\frac{8}{45} - b(x)\right)x^3 \tan x, \qquad (1)$$

where $a(x) = \frac{8}{945}x^2$, $b(x) = \frac{8}{945}x^2 - \frac{16}{14175}x^4$.

* Corresponding author at: Valahia University of Târgovişte, Bd. Unirii 18, 130082 Târgovişte, Romania. Tel.: +40 722727627.
E-mail addresses: maria.nenezic@gmail.com (M. Nenezić), malesevic@etf.rs (B. Malešević), cristinel.mortici@hotmail.com (C. Mortici).

http://dx.doi.org/10.1016/j.amc.2016.02.035
0096-3003/

Theorem 1.2. *For every* $x \in (\frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2})$ *in the left-hand side and for every* $x \in (\frac{\pi}{3} - \frac{1}{2}, \frac{\pi}{2})$ *in the right-hand side the following inequalities hold true*:

$$2 + \left(\frac{16}{\pi^4} + c(x)\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\frac{16}{\pi^4} + d(x)\right)x^3 \tan x, \tag{2}$$

where

$$c(x) = \left(\frac{160}{\pi^5} - \frac{16}{\pi^3}\right)\left(\frac{\pi}{2} - x\right), d(x) = \left(\frac{160}{\pi^5} - \frac{16}{\pi^3}\right)\left(\frac{\pi}{2} - x\right) + \left(\frac{960}{\pi^6} - \frac{96}{\pi^4}\right)\left(\frac{\pi}{2} - x\right)^2.$$

Theorems 1.1. and 1.2. describe a subtly analysis of Wilker inequality by Mortici. The method of proving the inequalites in this paper was presented in [5] and it is based on the use of appropriate approximations of some mixed trigonometric polynomials with finite Taylor series. This is a continuation of the method by Mortici presented in [4]. The method from [5] was applied in [6–8] for related inequalities.

2. The main results

The authors of this paper provide an automatization for proving of mixed trigonometric inequalities, where the original computation method was presented in Malešević and Makragić [5]. Mortici [1] made a subtly analysis in the sense that he looked for inequalities of higher precision. This paper shows that with minor modification of the functions $a(x)$, $b(x)$, $c(x)$ and $d(x)$ from Mortici [1] is possible to get more precise inequalities which hold on the whole interval $x \in (0, \pi/2)$.

The main purpose of our paper is to extend the intervals defined in theorems given by Mortici [4]. More precisely, we extend the domains $(0, 1)$ and $(\frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2})$ from the previous theorems to $(0, \frac{\pi}{2})$. We give the next two statements.

Theorem 2.1. *For every* $x \in (0, \frac{\pi}{2})$ *the following inequalities hold true:*

$$2 + \left(\frac{8}{45} - a(x)\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\frac{8}{45} - b_1(x)\right)x^3 \tan x,$$

where $a(x) = \frac{8}{945}x^2,\ b_1(x) = \frac{8}{945}x^2 - \frac{\boldsymbol{a}}{14175}x^4$ *with* $\boldsymbol{a} = \frac{480\pi^6 - 40320\pi^4 + 3628800}{\pi^8} = 17.15041\ldots$.

Theorem 2.2. *For every* $x \in \left(0, \frac{\pi}{2}\right)$ *the following inequalities hold true*:

$$2 + \left(\frac{16}{\pi^4} + c(x)\right)x^3 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \left(\frac{16}{\pi^4} + d(x)\right)x^3 \tan x,$$

where

$$c(x) = \left(\frac{160}{\pi^5} - \frac{16}{\pi^3}\right)\left(\frac{\pi}{2} - x\right),\ \ d(x) = \left(\frac{160}{\pi^5} - \frac{16}{\pi^3}\right)\left(\frac{\pi}{2} - x\right) + \left(\frac{960}{\pi^6} - \frac{96}{\pi^4}\right)\left(\frac{\pi}{2} - x\right)^2.$$

In [5] is considered a method for proving trigonometric inequalities for mixed trigonometric polynomials:

$$f(x) = \sum_{i=1}^{n} \alpha_i x^{p_i} \cos^{q_i} x \sin^{r_i} x > 0, \tag{3}$$

$x \in (\delta_2, \delta_1)$, $\delta_2 < 0 < \delta_1$, where $\alpha_i \in \mathbb{R} \setminus \{0\}$, $p_i, q_i, r_i \in \mathbb{N}_0$ and $n \in \mathbb{N}$. One method for proving inequalities of type (3) is based on transformation, using the sum of sine and cosine of multiple angles.

Let us mention some facts from [5]. Let $\varphi : [a, b] \longrightarrow \mathbb{R}$ be a function which is differentiable on $[a, b]$ and indefinitely differentiable on a right neighborhood of the point $x = a$. Denote by $T_m^{\varphi,a}(x)$ the Taylor polynomial of the function $\varphi(x)$ at $x = a$ of order m. Assume there is some $\eta > 0$ such that: $T_m^{\varphi,a}(x) \geq \varphi(x)$, for every $x \in (a, a+\eta) \subset [a, b]$; then define $\overline{T}_m^{\varphi,a}(x) = T_m^{\varphi,a}(x)$ and $\overline{T}_m^{\varphi,a}(x)$ present an upper approximation of $\varphi(x)$ on $(a, a+\eta)$ of a of order m. Analogously, if there is some $\eta > 0$ such that: $T_m^{\varphi,a}(x) \leq \varphi(x)$, $x \in (a, a+\eta) \subset [a, b]$; then let $\underline{T}_m^{\varphi,a}(x) = T_m^{\varphi,a}(x)$ and $\underline{T}_m^{\varphi,a}(x)$ present an under-approximation of $\varphi(x)$ on $(a, a+\eta)$ of a of order m. Note that we can define upper and under approximations on the left neighborhood of a point.

The following lemmas are proven in [5]:

Lemma 2.3. ***(i)*** *Let* $T_n(t) = \sum_{i=0}^{(n-1)/2} \frac{(-1)^i t^{2i+1}}{(2i+1)!}$, *where* $n = 4k+1$,

$k \in \mathbb{N}_0$. *Then:*

$$\left(\forall t \in \left[0, \sqrt{(n+3)(n+4)}\,\right]\right) \overline{T}_n(t) \geq \overline{T}_{n+4}(t) \geq \sin t, \tag{4}$$

$$\left(\forall t \in \left[-\sqrt{(n+3)(n+4)}, 0\right]\right) \underline{T}_n(t) \leq \underline{T}_{n+4}(t) \leq \sin t. \tag{5}$$