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Existence of solitary solutions in a class of nonlinear differential equations with polynomial nonlinearity

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ABSTRACT

The inverse balancing method for the determination of the necessary conditions of existence of solitary solutions to *m*th order differential equations with *n*th order polynomial nonlinearity is presented in this paper. It is shown that the order of possible solitary solutions does not increase if orders of the differential equation and the polynomial nonlinearity increase. Furthermore, the relationships between the order of the solitary solution and the order of the equation (and the nonlinearity) are given in the explicit form.

1. Introduction

With the growth of computational power, a number of methods based on symbolic computations for the construction of solutions to nonlinear differential equations have been developed during the recent decades. Special attention has been devoted to solitary solutions of differential equations with polynomial nonlinearity. Traveling waves in a one-dimensional model of hemodynamics are studied in [1]; the solitary solutions of the variant Boussinesq equations used in water wave modeling are considered in [2,3]. Four aspects of solitary wave solutions of high-level Green–Naghdi equations are discussed in [4]. Exact solitary solutions to the Kuramoto–Sivashinsky equation, which describes the fluctuation of the position of a flame front, are considered in [5] utilizing the consistent Riccati expansion method and in [6,7] with the tanh method. In [8], the $(\frac{G'}{G})$-expansion method has been applied to study solitary solutions of Fisher's equation, which describes the process of interaction between diffusion and reaction. The same method has been applied to the Klein–Gordon equation arising in quantum field theory [9]. The Exp-function method has been used to compute solitary solutions to the Dullin–Gottwald–Holm equation used in hydrodynamics [10] and the Cahn–Allen equation describing the process of phase separation in iron alloys [11].

The Exp-function method [12,13], the tanh-function method [14,15], the $(\frac{G'}{G})$ expansion method [16,17] are typical examples of techniques for the identification of closed-form solitary solutions to nonlinear evolutions in mathematical physics. However, straightforward application of these methods has attracted a considerable amount of criticism [18–20].

One of the main criticisms of the Exp-function method is that the obtained solitary solutions do not always satisfy the differential equation in the general case [18,20].

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A class of simplest equation methods for the determination of exact solutions to nonlinear differential equations, first introduced by Kudryashov in [21], does not possess the drawbacks of the Exp-function method. The basis of the simplest equation method is to use the solutions of the simplest nonlinear differential equations to express the solution of the given equation [21]. The simplest equation method has been extended and applied to the Sharma–Tasso–Olver and Burgers–Huxley equations in [22]. The exact solutions of a model describing the pattern formation processes on the semiconductor surfaces under ion beam bombardment are considered using the simplest equation method in [23]. The method has been generalized for application to non-autonomous differential equations and applied to the Painlevé equations in [24].

The modification of the Simplest equation method, due to Vitanov et al. in [25–27] is used to obtain exact traveling-wave solutions for two classes of model PDEs from ecology and population dynamics [28]. The modified simplest equation method has been demonstrated to yield solitary wave solutions for nonlinear partial differential equations in [29] and has been applied to compute traveling-wave solutions to the Swift–Hohenberg and generalized Rayleigh equations [26] as well as the generalized Kuramoto–Sivashinsky, reaction–diffusion equation with density-dependent diffusion, and the reaction-telegraph equations [25].

The main objective of this paper is to demonstrate an analytical framework based on the simplest equation method for the identification of solitary solutions to the following partial differential equation:

$$\frac{\partial^m u}{\partial t^m} + A_{m-1,0}\frac{\partial^{m-1} u}{\partial t^{m-1}} + A_{0,m-1}\frac{\partial^{m-1} u}{\partial z^{m-1}} + \cdots + A_{10}\frac{\partial u}{\partial t} + A_{01}\frac{\partial u}{\partial z} = a_n u^n + \cdots + a_0, \tag{1}$$

where $A_{j,r}, a_k \in \mathbb{R}$; $j, r = 1, \ldots, m-1$, $k = 0, \ldots, n$ and $a_n \neq 0$. The wave variable substitution $x := kt + \omega z$; $k, \omega \in \mathbb{R}$ transforms (1) to the m-th order differential equation with constant coefficients and nth order polynomial nonlinearity:

$$y_x^{(m)} + b_{m-1} y_x^{(m-1)} + \cdots + b_1 y_x' = a_n y^n + a_{n-1} y^{n-1} + \cdots + a_0, \tag{2}$$

with $b_j \in \mathbb{R}$, $j = 1, \ldots, m$.

The necessary conditions of existence of the solitary solution to (1):

$$y_0 = y_0(x) = \sigma \frac{\prod_{j=1}^{l} (e^{\eta(x-c)} - y_j)}{\prod_{j=1}^{l} (e^{\eta(x-c)} - x_j)}, \tag{3}$$

where $l \in \mathbb{N}, \sigma, \eta, c \in \mathbb{R}, \sigma, \eta \neq 0; y_j, x_j \in \mathbb{C}, j = 1, \ldots, l$ are derived in terms of the equation order n, m and the solution order l.

Nonlinear partial differential equations of the form (1) have already been considered in literature. Nonlinear equations with polynomial nonlinearity up to the fourth order that admit solitary solutions are discussed in [30,31]. A discussion of the existence of exact solutions for a seventh order nonlinear partial differential equation can be found in [32]. The paper [33] contains an extensive discussion on the solitary solutions of various well-known nonlinear partial differential equations that include the form of the solution (3) and variants of Eq. (1). Traveling wave solutions to Eq. (1) without mixed derivatives with nonlinearities up to the fifth order are derived using the modified simplest equation method in [34]. Polynomial nonlinearities in differential equations that model interacting populations are discussed in [35].

Our approach is to determine the parameters of the differential equation in terms of the parameters of the solution. This technique allows the determination of constraints on the order of the differential equation, the nonlinearity terms and the solitary solution for Eq. (1) to admit solitary solutions (3). It is also demonstrated that if the condition on n, m and l is satisfied, additional constraints on the parameters of (2) and (3) must be imposed to ensure the existence of (3) as a solution to (1).

2. Inverse balancing method

2.1. Simplification of (3)

The variable substitution $\hat{x} := e^{\eta(x-c)}$ is introduced. Then, (3) reads:

$$y_0(x) = \hat{y}_0(\hat{x}) = \sigma \frac{Y_l(\hat{x})}{X_l(\hat{x})}, \quad Y_l(\hat{x}) := \prod_{j=1}^{l} (\hat{x} - y_j), \quad X_l(\hat{x}) := \prod_{j=1}^{l} (\hat{x} - x_j). \tag{4}$$

Note that

$$y_x' = \eta \hat{x} \hat{y}_{\hat{x}}', \quad y_x^{(k)} = (\eta \hat{x} \hat{y}_{\hat{x}}')_{\hat{x}}^{(k-1)} = \eta^k \sum_{j=1}^{k} c_{kj} \hat{y}_{\hat{x}}^{(j)} \hat{x}^j, \quad k = 2, 3, \ldots, \tag{5}$$

where $c_{kj} \in \mathbb{R}$, $j = 1, \ldots, k$,. Then (2) reads:

$$\eta^m \hat{x}^m \hat{y}_{\hat{x}}^{(m)} + \hat{b}_{m-1} \hat{x}^{m-1} \hat{y}_{\hat{x}}^{(m-1)} + \cdots + \hat{b}_1 \hat{x} \hat{y}_{\hat{x}}' = a_n \hat{y}^n + a_{n-1} \hat{y}^{n-1} + \cdots + a_0. \tag{6}$$

The coefficients $\hat{b}_k$, $k = 1, \ldots, m-1$ are linear combinations of c_{kj}.