Applied Mathematics and Computation 283 (2016) 385–396

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Monotonicity properties, inequalities and asymptotic expansions associated with the gamma function

Chao-Ping Chen*

School of Mathematics and Informatics, Henan Polytechnic University, Jiaozuo 454000, Henan, China

ARTICLE INFO

MSC:
Primary 33B15
Secondary 41A60
26D15

Keywords:
Gamma function
Asymptotic expansion
Inequality

ABSTRACT

We define $\vartheta(x)$ by the equality

$$\Gamma(x+1)=\sqrt{\pi}\left(\frac{x}{e}\right)^x\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}e^{\vartheta(x)}.$$

We call $\vartheta(x)$ the remainder of Ramanujan's formula. In this paper, we present some properties for $\vartheta(x)$, including monotonicity properties, inequalities and asymptotic expansions. Furthermore, we present some full asymptotic expansions for the gamma function related to the Nemes, Ramanujan and Burnside formulas.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \in \mathbb{N} := \{1, 2, \dots\} \tag{1.1}$$

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Eq. (6.1.40)]) by

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right)\\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\cdots\right)\end{aligned} \tag{1.2}$$

* Tel.: +28 13603913986.
E-mail address: chenchaoping@sohu.com

http://dx.doi.org/10.1016/j.amc.2016.02.040

as $x \to \infty$, where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers (see, for example, [58, Section 1.7]). Stirling's formula has attracted much interest of several mathematicians and has motivated a large number of research papers concerning various generalizations and improvements (see [5,6,8–13,15–18,21,23–25,28–34,36–56] and the references cited therein). See also an overview in [35].

The Indian mathematician Ramanujan (see [57, p. 339] and [4, pp. 117–118]) proposed the claim that

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta_x}{30}\right)^{1/6},$$

where $\theta_x \to 1$ as $x \to \infty$ and $\frac{3}{10} < \theta_x < 1$. That is,

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6}, \quad x \to \infty \tag{1.3}$$

and

$$\sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{100}\right)^{1/6} < \Gamma(x+1) < \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6}, \quad x \geq 0.$$

Ramanujan's claim has been the subject of intense investigations and is reviewed in [7, p. 48, Question 754], and has motivated a large number of research papers (see, for example, [3,10,11,15,23–25,47,48,50]).

In 2001, Karatsuba [25] proved that the function

$$h(x) := \left[\left(\frac{e}{x}\right)^x \frac{\Gamma(x+1)}{\sqrt{\pi}}\right]^6 - (8x^3 + 4x^2 + x) = \frac{\theta_x}{30}$$

increases monotonically from $[1, \infty)$ onto $[h(1), h(\infty))$, where $h(1) = e^6/\pi^3 - 13 = 0.0111976\ldots$ and $h(\infty) = 1/30 = 0.0333\ldots$. Also, Karatsuba [25, Eq. (5.5)] established the asymptotic representation of the gamma function

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30} - \frac{11}{240x} + \frac{79}{3360x^2} + \frac{3539}{201600x^3} - \frac{9511}{403200x^4} - \frac{10051}{716800x^5} + \frac{233934691}{6386688000x^6} + \cdots\right)^{1/6}, \quad x \to \infty \tag{1.4}$$

(x^{-6} term corrected). Moreover, the author gave a formula for successively determining the coefficients. In 2003, Alzer [3] proved that in (0, 1] the constant term $\frac{1}{100}$ can be replaced by the best possible $\min_{0.6 \leq x \leq 0.7} \theta_x = 0.0100450\ldots$ and that the improved double inequality for θ_x holds for $0 \leq x < \infty$.

Recently, Mortici [47] obtained the following approximation formula:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} \times \exp\left(-\frac{11}{11520x^4} + \frac{13}{13440x^5} + \frac{1}{691200x^6} - \frac{421}{691200x^7}\right), \quad x \to \infty. \tag{1.5}$$

Motivated by (1.5), we define $\vartheta(x)$ by the equality

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{1}{30}\right)^{1/6} e^{\vartheta(x)}. \tag{1.6}$$

We call $\vartheta(x)$ the remainder of Ramanujan's formula.

The first aim of this paper is to present some properties for $\vartheta(x)$, including monotonicity properties, inequalities and asymptotic expansions.

A more accurate approximation than Stirling's formula (1.1) is Burnside's formula [9],

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}. \tag{1.7}$$

Using the continued fractions, Lu [34] provided an asymptotic expansion for the gamma function starting from Burnside's formula (1.7) as follows:

$$\Gamma(x+1) \sim \sqrt{2\pi}\left(\frac{x+\frac{1}{2}}{e}\right)^{x+\frac{1}{2}} \left(1 + \cfrac{a_1}{x^2 + \cfrac{a_2 x}{x + \cfrac{a_3 x}{x + \cfrac{a_4 x}{x + \ddots}}}}\right)^{x-\frac{1}{2}}, \tag{1.8}$$