Applied Mathematics and Computation 276 (2016) 1–11

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Two cubic spline methods for solving Fredholm integral equations☆

A. Bellour [a], D. Sbibih [b,*], A. Zidna [c]

[a] Department of Mathematics, Ecole Normale Supérieure de Constantine, Constantine, Algeria
[b] University Mohammed I, URAC05, FSO, MATSI Laboratory, Oujda, Morocco
[c] LITA Laboratory, University of Lorraine, Metz, France

ARTICLE INFO

PACS:
45L05
65R20
65D30
65L70

Keywords:
Cubic spline interpolation
Quasi-interpolation
Fredholm integral equation

ABSTRACT

In this work, we propose two methods based on the use of natural and quasi cubic spline interpolations for approximating the solution of the second kind Fredholm integral equations. Convergence analysis is established. Some numerical examples are given to show the validity of the presented methods.

1. Introduction

We are interested in the approximation of the solutions of the following Fredholm integral equations:

$$x(t) = f(t) + \int_a^b k(t,s)x(s)ds, t \in I = [a,b], \tag{1.1}$$

where the functions f and k are sufficiently smooth. The existence and the uniqueness of the smooth solution are given by many authors (see, for example, [14,15,20]).

The numerical solutions of Fredholm integral equations have been investigated by many authors (see, for example, [1,2,4,5,7–10,16]). The monograph [4] presents a historical survey of many collocation methods for (1.1).

Allouch et al. [1,2] proposed a numerical method by approximating the kernel k by using spline quasi-interpolants. Atkinson et al. [5] applied a continuous collocation method to approximate the solution of (1.1). Borzabadi and Fard [8] used a collocation iterative method to find a numerical solution of nonlinear Fredholm integral equations.

The natural and the quasi cubic spline interpolation for approximating the solution of integral equations, ordinary differential equations, partial differential equations have been proposed by many authors (see, for example, [6,11–13,19,21]).

This paper is concerned with the numerical solution of (1.1). We propose two main methods to find an approximate solution of (1.1) in the space $S_3^2(I, \Pi_n)$. More precisely, it is organized as follows. In Section 2, we describe the first collocation method which is based on the use of natural cubic spline interpolation, and we show that it has an approximation order $\mathcal{O}(h^4)$ on any compact subinterval included in $]a, b[$. In Section 3, we propose another method, based on cubic spline quasi-interpolation, which gives

☆ Research supported by URAC05.
* Corresponding author. Tel.: +212 671790072.
E-mail addresses: bellourazze123@yahoo.com (A. Bellour), sbibih@yahoo.fr (D. Sbibih), ahmed.zidna@univ-lorraine.fr (A. Zidna).

http://dx.doi.org/10.1016/j.amc.2015.11.055
0096-3003/

rise to two techniques for approximating the considered equation. We show that these both techniques have an approximation order $\mathcal{O}(h^4)$ on the whole interval $[a, b]$. In both cases, the approximation order four is reached under some conditions on the kernel k. Section 4 is devoted to three numerical examples that illustrate the theoretical results. Finally, in Section 5, we give a conclusion where we discuss a comparison between the two main methods.

2. Natural cubic spline approximation

Let $\Pi_n = \{t_i = a + ih, i = 0, 1, \ldots, n\}$ be a uniform partition of the interval I, where the stepsize $h = \frac{b-a}{n}$, and let $t_{-3} = t_{-2} = t_{-1} = t_0$, $t_{n+3} = t_{n+2} = t_{n+1} = t_n$. We denote by $S_3^2(I, \Pi_n)$ the space of C^2 cubic splines with knot set Π_n and multiple set at the endpoints. It has dimension $n+3$ and the cubic B-splines $\{B_i, i = 0, 1, \ldots, n+2\}$ with support $[t_{i-3}, t_{i+1}]$ form a basis of this space.

Let $S \in S_3^2(I, \Pi_n)$ be a cubic spline generated by boundary conditions (see [8]) interpolating the $(n+1)$ values $x_i = x(t_i)$, $i = 0, \ldots, n$, i.e. $S(t_i) = x_i$, with natural boundary conditions $S''(a) = S''(b) = 0$. Then, the restrictions of S to the intervals $\sigma_i = [t_i, t_{i+1}]$, $i = 0, \ldots, n-1$, can be written in the form:

$$S_i(t) = \frac{z_{i+1}}{6h}(t - t_i)^3 + \frac{z_i}{6h}(t_{i+1} - t)^3 + \left(\frac{x_{i+1}}{h} - \frac{h}{6}z_{i+1}\right)(t - t_i) + \left(\frac{x_i}{h} - \frac{h}{6}z_i\right)(t_{i+1} - t),$$

where $z_i = S''(t_i)$, $i = 0, 1, \ldots, n$.

The values z_i are determined as the solutions of the following linear system:

$$z_{i-1} + 4z_i + z_{i+1} = \frac{6}{h^2}(x_{i-1} - 2x_i + x_{i+1}), i = 1, \ldots, n-1,$$

with $z_0 = S''(a) = z_n = S''(b) = 0$.

In the matrix notation, the above system has the form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & \ldots & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ \vdots \\ \vdots \\ z_{n-2} \\ z_{n-1} \\ z_n \end{pmatrix} = \frac{6}{h^2} \begin{pmatrix} 0 \\ x_0 - 2x_1 + x_2 \\ x_1 - 2x_2 + x_3 \\ \vdots \\ \vdots \\ x_{n-3} - 2x_{n-2} + x_{n-1} \\ x_{n-2} - 2x_{n-1} + x_n \\ 0 \end{pmatrix} \tag{2.1}$$

It is well known, see [3], that for x, solution of the Eq. (1.1), smooth enough and for all $t \in J = [c, d] \subset [a, b]$ we have $\|x - S\|_{\infty,J} = \mathcal{O}(h^4)$. Hence, from (1.1), we obtain for all $i = 0, \ldots, n$

$$\begin{aligned} x_i &= f_i + \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} k(t_i, r) s_j(r) dr + \mathcal{O}(h^4) \\ &= f_i + \sum_{j=0}^{n-1} \frac{z_{j+1}}{6h} \underbrace{\int_{t_j}^{t_{j+1}} k(t_i, r)(r - t_j)^3 dr}_{a_{i,j+1}} + \sum_{j=0}^{n-1} \frac{z_j}{6h} \underbrace{\int_{t_j}^{t_{j+1}} k(t_i, r)(t_{j+1} - r)^3 dr}_{b_{i,j}} \\ &\quad + \sum_{j=0}^{n-1} \left(\frac{x_{j+1}}{h} - \frac{h}{6} z_{j+1}\right) \underbrace{\int_{t_j}^{t_{j+1}} k(t_i, r)(r - t_j) dr}_{c_{i,j+1}} + \sum_{j=0}^{n-1} \left(\frac{x_j}{h} - \frac{h}{6} z_j\right) \underbrace{\int_{t_j}^{t_{j+1}} k(t_i, r)(t_{j+1} - r) dr}_{d_{i,j}} + \mathcal{O}(h^4) \\ &= f_i + \frac{1}{6h} \sum_{j=0}^{n} z_j a_{i,j} + \frac{1}{6h} \sum_{j=0}^{n} z_j b_{i,j} - \frac{h}{6} \sum_{j=0}^{n} z_j c_{i,j} + \frac{1}{h} \sum_{j=0}^{n} x_j c_{i,j} + \frac{1}{h} \sum_{j=0}^{n} x_j d_{i,j} - \frac{h}{6} \sum_{j=0}^{n} z_j d_{i,j} + \mathcal{O}(h^4), \end{aligned} \tag{2.2}$$

such that, $a_{i,0} = b_{i,n} = c_{i,0} = d_{i,n} = 0$ for $i = 0, \ldots, n$.

Now, we approximate x_i by $\widehat{x}_i$ and z_i by $\widehat{z}_i$ such that $\widehat{x}_i$ and $\widehat{z}_i$, $i = 0, \ldots, n$, satisfy the system (2.1) and for all $i = 0, \ldots, n$, we have

$$\widehat{x}_i = f_i + \frac{1}{6h} \sum_{j=0}^{n} \widehat{z}_j a_{i,j} + \frac{1}{6h} \sum_{j=0}^{n} \widehat{z}_j b_{i,j} - \frac{h}{6} \sum_{j=0}^{n} \widehat{z}_j c_{i,j} + \frac{1}{h} \sum_{j=0}^{n} \widehat{x}_j c_{i,j} + \frac{1}{h} \sum_{j=0}^{n} \widehat{x}_j d_{i,j} - \frac{h}{6} \sum_{j=0}^{n} \widehat{z}_j d_{i,j}.$$