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On (p, q)-analogue of Kantorovich type Bernstein–Stancu–Schurer operators

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ABSTRACT

In this paper, we introduce a new kind of Kantorovich-type Bernstein–Stancu–Schurer operators based on the concept of (p, q)-integers. We investigate statistical approximation properties and establish a local approximation theorem, we also give a convergence theorem for the Lipschitz continuous functions. Finally, we give some graphics and numerical examples to illustrate the convergence properties of operators to some functions.

1. Introduction

Recently, Mursaleen et al. applied (p, q)-calculus in approximation theory and introduced the (p, q)-analogue of Bernstein operators in [1]. We mention some of their other works as [2–6].

In this paper, we will introduce a new kind of Kantorovich-type Bernstein–Stancu–Schurer operators based on (p, q)-integers which will be defined in (1), we also investigate some approximation properties and give some graphics and numerical examples to illustrate the convergence to some functions.

Before introducing the operators, we mention certain definitions based on (p, q)-integers, details can be found in [7–11]. For any fixed real number $0 < q < p \leq 1$ and each nonnegative integer k, we denote (p, q)-integers by $[k]_{p,q}$, where

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}.$$

Also (p, q)-factorial and (p, q)-binomial coefficients are defined as follows:

$$[k]_{p,q}! = \begin{cases} [k]_{p,q}[k-1]_{p,q}\dots[1]_{p,q}, & k = 1, 2, \dots; \\ 1, & k = 0, \end{cases} \qquad \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!},$$

where $n \geq k \geq 0$. The (p, q)-Binomial expansion is defined by

$$(x+y)^n_{p,q} = \begin{cases} 1, & n = 0; \\ (x+y)(px+qy)\dots\left(p^{n-1}x + q^{n-1}y\right), & n = 1, 2, \dots \end{cases}.$$

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The definite (p, q)-integrals are defined by

$$\int_0^a f(x)d_{p,q}x = (q-p)a\sum_{k=0}^{\infty}\frac{p^k}{q^{k+1}}f\left(\frac{p^k}{q^{k+1}}a\right), \quad if\left|\frac{p}{q}\right| < 1,$$
$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{k=0}^{\infty}\frac{q^k}{p^{k+1}}f\left(\frac{q^k}{p^{k+1}}a\right), \quad if\left|\frac{p}{q}\right| > 1.$$

When $p = 1$, all the definitions of (p, q)-calculus above are reduced to q-calculus.

For $f \in C(I)$, $I = [0, 1 + l]$, $l \in \mathbb{N}_0$, $0 \le \alpha \le \beta$, $0 < q < p \le 1$ and $n \in \mathbb{N}$, we introduce the (p, q)-analogue of Kantorovich-type Bernstein–Stancu–Schurer operators as follows:

$$K_{n,p,q}^{\alpha,\beta,l}(f;x) = ([n+1]_{p,q}+\beta)\sum_{k=0}^{n+l}\frac{b_{n+l,k}(p,q;x)}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}+\alpha}{[n+1]_{p,q}+\beta}}^{\frac{[k+1]_{p,q}+\alpha}{[n+1]_{p,q}+\beta}} f(t)d_{p,q}t, \tag{1}$$

where

$$b_{n+l,k}(p,q;x) = \begin{bmatrix} n+l \\ k \end{bmatrix}_{p,q} x^k(1-x)_{p,q}^{n+l-k}. \tag{2}$$

2. Auxiliary results

In order to obtain the approximation properties, We need the following lemmas.

Lemma 2.1. *For the (p, q)-analogue of Kantorovich-type Bernstein–Stancu–Schurer operators (1), we have*

$$K_{n,p,q}^{\alpha,\beta,l}(1;x) = 1, \tag{3}$$

$$K_{n,p,q}^{\alpha,\beta,l}(t;x) = \frac{(p+1)[n+l]_{p,q}x + (qx+1-x)_{p,q}^{n+l} + 2\alpha}{[2]_{p,q}([n+1]_{p,q}+\beta)}, \tag{4}$$

$$\begin{aligned} K_{n,p,q}^{\alpha,\beta,l}(t^2;x) = {} & \frac{(p^3+p^2+p)[n+l]_{p,q}[n+l-1]_{p,q}}{[3]_{p,q}([n+1]_{p,q}+\beta)^2}x^2 + \frac{3\alpha(p+1)[n+l]_{p,q}x}{[3]_{p,q}([n+1]_{p,q}+\beta)^2} \\ & + \frac{(p^2+p+1+2pq+q)(qx+1-x)_{p,q}^{n+l-1}[n+l]_{p,q}x}{[3]_{p,q}([n+1]_{p,q}+\beta)^2} \\ & + \frac{3\alpha(qx+1-x)_{p,q}^{n+l} + (q^2x+1-x)_{p,q}^{n+l} + 3\alpha^2}{[3]_{p,q}([n+1]_{p,q}+\beta)^2}. \end{aligned} \tag{5}$$

Proof. (3) is easily obtained from (1) and the definition of (p, q)-integrals. Using (1) and $[k+1]_{p,q} = q^k + p[k]_{p,q}$, we have

$$\begin{aligned} K_{n,p,q}^{\alpha,\beta,l}(t;x) &= ([n+1]_{p,q}+\beta)\sum_{k=0}^{n+l}\frac{b_{n+l,k}(p,q;x)}{[k+1]_{p,q}-[k]_{p,q}}\int_{\frac{[k]_{p,q}+\alpha}{[n+1]_{p,q}+\beta}}^{\frac{[k+1]_{p,q}+\alpha}{[n+1]_{p,q}+\beta}} t d_{p,q}t \\ &= ([n+1]_{p,q}+\beta)\sum_{k=0}^{n+l}\frac{b_{n+l,k}(p,q;x)}{[k+1]_{p,q}-[k]_{p,q}}\frac{([k+1]_{p,q}+\alpha)^2-([k]_{p,q}+\alpha)^2}{[2]_{p,q}([n+1]_{p,q}+\beta)^2} \\ &= \frac{1}{[2]_{p,q}([n+1]_{p,q}+\beta)}\sum_{k=0}^{n+l} b_{n+l,k}(p,q;x)([k+1]_{p,q}+[k]_{p,q}+2\alpha) \\ &= \frac{1}{[2]_{p,q}([n+1]_{p,q}+\beta)}\sum_{k=0}^{n+l} b_{n+l,k}(p,q;x)(qx)^k(1-x)_{p,q}^{n+l-k} \\ &\quad + \frac{p+1}{[2]_{p,q}([n+1]_{p,q}+\beta)}\sum_{k=0}^{n+l} b_{n+l,k}(p,q;x)[k]_{p,q} + \frac{2\alpha}{[2]_{p,q}([n+1]_{p,q}+\beta)} \\ &= \frac{(qx+1-x)_{p,q}^{n+l}}{[2]_{p,q}([n+1]_{p,q}+\beta)} + \frac{(p+1)[n+l]_{p,q}x}{[2]_{p,q}([n+1]_{p,q}+\beta)} + \frac{2\alpha}{[2]_{p,q}([n+1]_{p,q}+\beta)}. \end{aligned}$$