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Note on some periodic optimal harvesting problems for age-structured population dynamics

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ABSTRACT

We consider two optimal harvesting problems related to age-structured population dynamics with logistic term and time-periodic (of period T) control and vital rates. We derive the first order necessary optimality conditions for both problems. We prove that in the particular case of time-independent vital rates and under a natural hypothesis on the logistic term, the optimal control for the first problem is constant. The second problem concerns the constant harvesting efforts which act on subintervals of $[0, T]$. The maximizing here is treated with respect to the subinterval where the control acts. We find the directional derivative of the cost functional. Some final comments concerning the population dynamics with diffusion are given as well.

1. Introduction and setting of the problems

The present paper concerns two optimal harvesting problems related to some age-structured population dynamics with logistic term and time-periodic vital rates. The starting point is the following bilinear control problem which describes the dynamics of a harvested population

$$\begin{cases} \partial_t p + \partial_a p + \mu(a,t)p + \mathcal{M}(\Gamma(t))p = -u(t)p, & (a,t) \in Q \\ \Gamma(t) = \int_0^A \gamma(a)p(a,t)da, & t \in \mathbf{R}^+ \\ p(0,t) = \int_0^A \beta(a,t)p(a,t)da, & t \in \mathbf{R}^+ \\ p(a,0) = p_0(a), & a \in [0,A), \end{cases} \tag{1}$$

where $Q = [0, A) \times \mathbf{R}^+$ and $A \in (0, +\infty)$ is the maximal age of the population species. Here, $p(a, t)$ is the population density for age a and time t, β is the natural fertility rate and μ is the natural mortality rate, both time-periodic of period $T \in (0, +\infty)$ and $p_0(a)$ is the initial density of the population of age a; $\gamma(a)$ is the biomass of an individual of age a and $\mathcal{M}(\Gamma(t))$ represents an additional mortality rate due to the competition for resources. $u(t)$ is a time-periodic (of period T) harvesting effort (control).

The following natural hypotheses will used all over the present paper:

$$\beta \in C(\mathbf{R}^+; L^\infty(0,A)), \ \ \beta(a,t) \geq 0 \quad a.e.\ (a,t) \in Q,$$
$$\beta(a,t) = \beta(a,t+T) \quad a.e.\ (a,t) \in Q;$$

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$$\exists\delta, \tau > 0 \text{ and } a_0 \in (0, A) \text{ such that}$$
$$a_0 + T \le A \text{ and } \beta(a, \tau) \ge \delta \quad a.e.\ a \in (a_0, a_0 + T);$$
$$p_0 \in L^\infty(0, A),\ p_0(a) > 0 \quad a.e.\ a \in (0, A);$$
$$\gamma \in L^\infty(0, A),\ \gamma(a) \ge \gamma_0 > 0 \quad a.e.\ a \in (0, A).$$

We assume in addition that $\mathcal{M} : \mathbf{R}^+ \to \mathbf{R}^+$ is a continuously differentiable function with $\mathcal{M}'(r) > 0$ for any $r > 0$, $\mathcal{M}(0) = 0$ and $\lim_{r\to+\infty} \mathcal{M}(r) = +\infty$.

Denote by

$$\mathcal{U} = \{u \in L^\infty(\mathbf{R}^+);\ u(t) = u(t + T),\ 0 \le u(t) \le U \quad a.e.\ t \in \mathbf{R}^+\}.$$

Here, $U \in (0, +\infty)$ is the maximal value of the harvesting effort.

An ergodicity result established in [1] (see also [2]) implies that there exists a unique pair $(\alpha, q) \in \mathbf{R} \times C(\mathbf{R}^+; L^\infty(0, A))$ such that q is a nonnegative solution to

$$\begin{cases} \partial_t q + \partial_a q + \mu(a, t)q + \alpha q = 0, & (a, t) \in Q \\ q(0, t) = \int_0^A \beta(a, t)q(a, t)da, & t \in \mathbf{R}^+ \\ q(a, t) = q(a, t + T), & (a, t) \in Q, \end{cases} \tag{2}$$

which satisfies in addition

$$Ess\ sup\{q(a, t);\ (a, t) \in Q\} = 1.$$

Actually, the set of all solutions to (2) is a linear space of dimension one.

As in [3,4] it follows that for any $u \in \mathcal{U}$:

$$\lim_{t\to+\infty} \|p^u(t) - \tilde{p}^u(t)\|_{L^\infty(0,A)} = 0,$$

where p^u is the unique solution to (1) and $\tilde{p}^u$ is the biggest solution to

$$\begin{cases} \partial_t p + \partial_a p + \mu(a, t)p + \mathcal{M}(\Gamma(t))p = -u(t)p, & (a, t) \in Q \\ \Gamma(t) = \int_0^A \gamma(a)p(a, t)da, & t \in \mathbf{R}^+ \\ p(0, t) = \int_0^A \beta(a, t)p(a, t)da, & t \in \mathbf{R}^+ \\ p(a, t) = p(a, t + T), & (a, t) \in Q. \end{cases} \tag{3}$$

Moreover,

If $T\alpha > \int_0^T u(s)ds$, then $\tilde{p}^u$ is the unique nontrivial nonnegative solution to (3).

If $T\alpha \le \int_0^T u(s)ds$, then $\tilde{p}^u \equiv 0$ is the unique nonnegative solution to (3).

The first situation corresponds to a population which explode if the logistic term would not appear and the second one to a population which will remain beyond a certain value.

For the definition and basic properties of the solutions to (1), (2) and (3) we refer to [3–5]. Actually, problem (3) has at most two nonnegative solutions and admits the trivial solution.

The asymptotic behavior of the solution to (1) for the particular case of time-independent vital rates and for $u \equiv 0$ has been established in [6]. For other results concerning the periodic solutions for age-structured population dynamics we refer to [7–10].

The harvested biomass per unit at moment t, corresponding to the effort u, is

$$\Gamma^u(t) = \int_0^A \gamma(a)p^u(a, t)da$$

for (1) and

$$\tilde{\Gamma}^u(t) = \int_0^A \gamma(a)\tilde{p}^u(a, t)da$$

for (3). It follows immediately that

$$\lim_{t\to+\infty} |\Gamma^u(t) - \tilde{\Gamma}^u(t)| = 0.$$

Let $c \in (0, +\infty)$ be the cost of harvesting effort per unit. For any $u \in \mathcal{U}$ we have that

$$\left| \int_t^{t+T} (\Gamma^u(s) - c)u(s)ds - \int_t^{t+T} (\tilde{\Gamma}^u(s) - c)u(s)ds \right|$$
$$\le U \int_t^{t+T} |\Gamma^u(s) - \tilde{\Gamma}^u(s)|ds \le UT \sup_{s\ge t} |\Gamma^u(s) - \tilde{\Gamma}^u(s)|,$$