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Asymptotical formulas for Gaussian and generalized hypergeometric functions

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ABSTRACT

In this paper, we present several generalizations and refinements for the asymptotic formulas of Gaussian and generalized hypergeometric functions.

1. Introduction

1.1. Gaussian hypergeometric function $_2F_1$

For real numbers a, b, and c with $c \neq 0, -1, -2, \ldots,$ the Gaussian hypergeometric function $_2F_1$ is defined by

$$_2F_1(a, b; c; x) = F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

for $|x| < 1$, where the Pochhammer symbol

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

for $n = 1, 2, \ldots,$ and $(a)_0 = 1$ for $a \neq 0$, $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ $(\Re x > 0)$ is the gamma function. It is well known that the Gaussian hypergeometric function has many important applications in geometric function theory, number theory and several other contexts, and a lot of special functions and elementary functions are the particular cases or limiting cases. Especially, in 1980s, de Branges used hypergeometric functions to prove the famous Bieberbach conjecture, which has given function theorists a renewed interest to study the role of Gaussian hypergeometric function. For the above, and more properties of $F(a, b; c; x)$, see [1,3,5–8,10,11,13,16,19,21,24,25].

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The function $F(a, b; c; x)$ is said to be zero-balanced if $c = a + b$. In the zero-balanced case, there is a logarithmic singularity at $x = 1$, and Gauss proved the asymptotic formula (see [14])

$$F(a, b; a+b; x) \sim -\frac{1}{B(a, b)} \log(1-x), \quad x \to 1, \tag{1.1}$$

where

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \quad \Re(z) > 0, \quad \Re(w) > 0 \tag{1.2}$$

is the classical Beta function.

Ramanujan found a much sharper asymptotic formula (see [14])

$$B(a, b)F(a, b; a+b; x) + \log(1-x) = R(a, b) + O((1-x)\log(1-x)), \quad x \to 1, \tag{1.3}$$

where

$$R(a, b) = -\psi(a) - \psi(b) - 2\gamma, \tag{1.4}$$

$\psi(z) = \Gamma'(z)/\Gamma(z)$, $\Re(z) > 0$ and γ is the Euler–Mascheroni constant.

In order to refine Gauss' asymptotic formula (1.1), Anderson et al. [4] considered the following Problem 1.1.

Problem 1.1. Is it true that the function

$$x \to F(a, b; a+b; x) + \frac{1}{B(a, b)} \frac{1}{x} \log(1-x),$$

is monotone on $(0, 1)$ for suitable a and b?

The above function was shown to be monotone for $a, b \in (0, 1)$ or $(a, b) \in (1, \infty)$ in [2]. Indeed, they proved that

Theorem 1.1 ([[2], Theorem 1.4]). *Let $a, b \in (0, \infty)$, $B = B(a, b)$, $R = R(a, b)$, $x \in (0, 1)$ and*

$$f(x) = \frac{xF(a, b; a+b; x)}{\log[1/(1-x)]}.$$

Then the following statement are true:

1. *f is decreasing from $(0, 1)$ onto $(1/B, 1)$ if $a, b \in (0, 1)$;*
2. *f is increasing from $(0, 1)$ onto $(1, 1/B)$ if $a, b \in (1, \infty)$;*
3. *$f(x) \equiv 1$ for all $x \in (0, 1)$ if $a = b = 1$;*
4. *The function $g(x) = BF(a, b; a+b; x) + (1/x)\log(1-x)$ is increasing from $(0, 1)$ onto $(B-1, R)$ if $a, b \in (0, 1)$;*
5. *The function $g(x) = BF(a, b; a+b; x) + (1/x)\log(1-x)$ is decreasing from $(0, 1)$ onto $(R, B-1)$ if $a, b \in (1, \infty)$.*

Later, Qiu and Vuorinen [20] proved the following Theorems 1.2 and 1.3, where Theorem 1.2 extended the parts (4) and (5) in Theorem 1.1, while Theorem 1.3 is another refinement of Gauss' asymptotic formula (1.1).

Theorem 1.2 ([[20], Theorem 1.4]). *Let $a, b \in (0, \infty)$, $c = a + b$, $a_1^* = 1 - ab$, $a_2^* = 2ab - a - b$, $a_3^* = |a_1^*| + |a_2^*|$, $B = B(a, b)$, $R = R(a, b)$, $x \in (0, 1)$*

$$g(x) = BF(a, b; c; x) + \frac{1}{x}\log(1-x).$$

Then we have

1. *$g(x) \equiv 0$ for all $x \in (0, 1)$ if $a_3^* = 0$;*
2. *g is strictly increasing from $(0, 1)$ onto $(B-1, R)$ if $a_3^* \neq 0$ and $a_1^* \geq \max\{0, a_2^*\}$;*
3. *g is strictly decreasing from $(0, 1)$ onto $(R, B-1)$ if $a_3^* \neq 0$ and $a_1^* \leq \min\{0, a_2^*\}$;*
4. *In the other case not stated in parts (1)-(3), that is $a_2^* < a_1^* < 0$, g is not always monotone on $(0, 1)$.*

Theorem 1.3 ([[20], Theorem 1.5]). *Let $a, b \in (0, \infty)$, $A_1^* = A_1^*(a, b) = a + b + ab - 3$, $A_2^* = A_2^*(a, b) = a + b - 3ab + 1$, $A^* = |A_1^*| + |A_2^*|$, $B = B(a, b)$, $R = R(a, b)$, $r \in (0, 1)$ and*

$$h(r) = \frac{BF(a, b; a+b; r) + \log(1-r) - R}{[(1-r)/r]\log[1/(1-r)]}.$$

Then we have

1. *$h(r) \equiv 1$ for all $r \in (0, 1)$ if $A^* = 0$;*
2. *h is strictly decreasing from $(0, 1)$ onto $(ab, B-R)$ if $A^* \neq 0$ and $A_1^* \leq \min\{0, A_2^*\}$. In particular, with this condition, for all $r \in (0, 1)$,*

$$\begin{aligned} ab\frac{1-r}{r}\log\left(\frac{1}{1-r}\right) &< BF(a, b; a+b; r) + \log(1-r) - R \\ &< (B-R)\frac{1-r}{r}\log\left(\frac{1}{1-r}\right); \end{aligned}$$