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A factorization theorem for operators occurring in the Stokes, Brinkman and Oseen equations

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ABSTRACT

In many physical problems one is faced with solving partial differential equations of the form $L_1(L_1 + L_2)u = 0$, where L_1 and L_2 are linear operators. It is found in many cases that the solution u is of the form $u_1 + u_2$ where $L_1u_1 = 0$ and $(L_1 + L_2)u_2 = 0$. In this paper we present sufficient conditions under which such a splitting is possible. Moreover, we give explicit formulae for u_1 and u_2 for a given u. We also show in some examples where the operators satisfy the sufficient conditions and such a splitting is used extensively. In particular, we find a class of solutions for the unsteady Brinkman and unsteady Oseen equations using the splitting that we propose.

1. Introduction

A Stokes flow represents motion of a viscous fluid at low Reynolds number where the convection terms are too small to take into account. The Reynolds numbers associated with swimming bacteria, spermatozoids are very low (see [10,22,28]). This suggests that the flow generated by a swimming microorganism satisfies the Stokes equations (see [8]). There are uncanny number of research articles and monographs in the literature about the steady/unsteady Stokes equations (see [5,12,20,21] and the references given there). Singular solutions of unsteady Stokes flows in two dimensions are studied in [25]. In [11], the author has constructed the potential theory for two dimensional convex domains to prove existence of the unsteady Stokes equations. In [23], single and double layer potentials are used in order to study the existence and uniqueness of the solutions of the boundary value problems for the unsteady Stokes equations in Lipschitz domains. For integral equation methods to study these equations see [6,7].

Another important system of equations we consider in this paper is the system of Oseen equations. The steady Oseen models have attracted attention of many mathematicians (see [3,5,9]). In [4], the author has used Galerkin's method to prove existence of weak solutions, to the steady Oseen equations, which belong to homogeneous Sobolev spaces. For more general results in the existence and uniqueness theory of weak solutions to the Oseen equations readers can refer to [1,5]. For generalized fundamental solutions for the unsteady Oseen, unsteady Stokes equations see [2,24].

On the other hand, it can be found in the literature on the Brinkman and unsteady Stokes flows (see [16,18]) that one seeks solutions for the following partial differential equations: $\Delta(\Delta - k^2)u = 0$, $\Delta(\Delta - \frac{\partial}{\partial t})u = 0$. It is found that u can be written as $u = u_1 + u_2$ where u_1 satisfies $\Delta u_1 = 0$, and u_2 satisfies $(\Delta - k^2)u_2 = 0$ in the case of the Brinkman equations and $(\Delta - \frac{\partial}{\partial t})u_2 = 0$ in the case of the unsteady Stokes equations. This kind of splitting is an important step in finding the complete general solutions of the Brinkman equations and unsteady Stokes equations (see [17,19]). After such a kind of splitting, the general solution of $Lv = 0$

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can be obtained using the method of separation of variables where $L = \Delta$ or $\Delta - k^2 Id$ or $\Delta - \frac{\partial}{\partial t}$. The method to write a solution of the given linear equation of higher order as a sum of solutions of two auxiliary equations of lower order is due to the Swedish theoretical physicist Oseen (see [13]).

We would like to mention that the fluid dynamics problems that we consider in this article have two important properties. First, they correspond to flows at low Reynolds number so that we can neglect the convection term in the Navier–Stokes equations. This enables us to obtain the analytical solution in most of the cases. Second, they involve differential operators with constant coefficients which allow us to decouple the system of equations that represent the flow. In this article, we are interested in finding sufficient conditions under which splitting is possible for the solutions of the equations of the type $L_1(L_1 + L_2)u = 0$, where L_1, L_2 are linear operators. Further, we find u_1 and u_2 for a given u such that $L_1 u_1 = 0,\ (L_1 + L_2)u_2 = 0$.

The organization of this article is as follows. In Section 2, we state and prove our main result. In Section 3, we present some examples in which one can perform a splitting using our method. These examples are of two types. In the first category, we mention some cases in which the splitting results are known in the literature. In the next category, we give a couple of cases where the results are new. In all these cases we give explicit expressions for u_1 and u_2 in terms of u. We summarize our conclusions in Section 4.

2. Main results

Let L_1, L_2 be two linear operators on a normed linear space X such that L_2 is invertible.

Theorem 1. *For any given u such that $L_1(L_1 + L_2)u = 0$, if L_2^{-1} exists and commutes with L_1, i.e., $L_1 L_2^{-1} = L_2^{-1} L_1$, then there exists $u_1, u_2 \in X$ such that $u = u_1 + u_2$ and $L_1 u_1 = 0,\ (L_1 + L_2)u_2 = 0$.*

Proof. Choose $u_1 = L_2^{-1}(L_1 + L_2)u$. Then we have

$$L_1 u_1 = L_1 L_2^{-1}(L_1 + L_2)u = L_2^{-1}(L_1(L_1 + L_2)u) = 0.$$

A straightforward computation would give us

$$\begin{aligned}(L_1 + L_2)(u - u_1) &= (L_1 + L_2)u - (L_1 + L_2)u_1 \\ &= (L_1 + L_2)u - (L_1 + L_2)L_2^{-1}(L_1 + L_2)u \\ &= (L_1 + L_2)u - L_1 L_2^{-1}(L_1 + L_2)u - (L_1 + L_2)u \\ &= -L_2^{-1} L_1(L_1 + L_2)u = 0.\end{aligned}$$

Now we define $u_2 = u - u_1$. This proves our assertion. □

Even if L_2 is not an injective operator, by slightly modifying the proof of Theorem 1, it is possible to have the splitting result. This is given in the following theorem.

Theorem 2. *Assume that L_2 is not an injective map. Further, assume that there exists at least one element u_1 in the set $\{v | L_2 v = (L_1 + L_2)u\}$ such that $L_1 u_1 = 0$, then we have $(L_1 + L_2)(u - u_1) = 0$.*

Proof. Consider

$$(L_1 + L_2)(u - u_1) = (L_1 + L_2)u - (L_1 u_1 + L_2 u_1) = (L_1 + L_2)u - L_2 u_1 = 0.$$

The last equality follows from the assumptions on u_1. □

Therefore, in this case also we are able to write u as sum of u_1 and u_2 with $L_1 u_1 = 0,\ (L_1 + L_2)u_2 = 0$.

Remark 1. Unlike in Theorem 1, we have not assumed that $L_1 L_2^{-1} = L_2^{-1} L_1$ in Theorem 2.

Remark 2. It is easy to see that if L_2 is invertible then, $L_1 L_2 = L_2 L_1$ if and only if $L_1 L_2^{-1} = L_2^{-1} L_1$.

Remark 3. Converse of Theorem 1 is true in the following sense.
If $L_1 u_1 = 0,\ (L_1 + L_2)u_2 = 0$ and $L_1 L_2 = L_2 L_1$ then $L_1(L_1 + L_2)(u_1 + u_2) = 0$.

3. Applications

In this section we give five applications of Theorems 1, and 2. For the derivation of the equations that we will consider in the following examples see, for instance, [5,12,21]. In all the examples that we present here, the pressure term is eliminated from the equations of motion resulting increase in the order of the equations which the velocity term satisfies (see [17–19,25,27]). However, these higher order equations can be solved with same boundary conditions. For details readers may refer [14–16,18].

Example 1 (The steady Brinkman equations)**.** Consider the steady Brinkman equations

$$-\nabla p + \mu\left(\Delta - \frac{1}{k}\right)\mathbf{q} = 0, \tag{1}$$