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Fractal dimension of random attractors for stochastic non-autonomous reaction–diffusion equations

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ABSTRACT

In this paper, we first give some conditions for bounding the fractal dimension of a random invariant set for a non-autonomous random dynamical system on a separable Banach space. Then we apply these conditions to prove the finiteness of fractal dimension of the random attractors for stochastic reaction–diffusion equations with multiplicative white noise and additive white noise.

1. Introduction

Recently, there have many publications concerning the random attractors for the infinite dimensional autonomous and non-autonomous random dynamical systems and applications to stochastic evolution equations, see [1–29] and the references therein. As we know, the finite dimensionality of an attractor is an important topic in describing the asymptotic behavior of infinite-dimensional dynamical systems [21–35]. Until now, there have several useful methods to estimate the upper bound of the Hausdorff and fractal dimensions of a random attractor, see [24–29]. For example, Crauel and Flandoli [24], Debussche [25,26] and Langa and Robinson [28] developed some technique to estimate the bound of Hausdorff and fractal dimensions of a random attractor for an autonomous random dynamical system basing on the differentiability of the system; Wang and Tang [29] gave a method to estimate the fractal dimension of a random attractor similar to [28], but requiring that the “Lipschitz constant” of the system and the “contraction coefficient” of the infinite-dimensional part of the system are both uniformly bounded with respect to the sample points, which are “strong” and satisfied only by some special random systems with uniform bounded derivative of the nonlinearity. However, in general, a random attractor for a nonlinear random system is not uniformly bounded along the sample path of a sample point which is the main difference from a global attractor for a deterministic system.

It is well known from [36–38] that if a compact set A in a metric space has a bounded fractal dimension $\dim_f(A) < m/2$ for some $m \in \mathbb{N}$, then A can be placed in the graph of a Hölder continuous mapping which maps a compact subset of $\mathbb{R}^m$ onto A. This implies that the finiteness of fractal dimension of an attractor plays a very important role in the finite-dimensional reduction theory of an infinite dimensional system. But just knowing the boundedness of Hausdorff dimension of an attractor for a system, we still have no available finite parameterization (see [39]).

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Motivated by the ideas of Wang [12], Debussche [25], Langa and Robinson [28] and Wang and Tang [29], in this article, we will do the following two things.

(I) We give some conditions for bounding the fractal dimension of a random invariant set for a non-autonomous random dynamical system originated from Debussche [25]. Our conditions do not require the differentiability of the random dynamical system and just need the boundedness of expectation for some random variables.

(II) As applications, we consider the finiteness of fractal dimension of the random attractors for the following stochastic non-autonomous reaction–diffusion equation with multiplicative white noise

$$\begin{cases} du - \Delta u dt + f(u)dt = g(x,t)dt + bu \circ dW(t) \quad \text{in} \quad U \times (\tau, +\infty), \quad \tau \in \mathbb{R}, \\ u(x,t)|_{x \in \partial U} = 0, \quad t \geq \tau; \quad u(x,\tau) = u_\tau(x), \quad x \in U, \end{cases} \tag{1.1}$$

and with additive white noise

$$\begin{cases} du - \Delta u dt + f(u)dt = g(x,t)dt + h(x)dW(t) \quad \text{in} \quad U \times (\tau, +\infty), \quad \tau \in \mathbb{R}, \\ u(x,t)|_{x \in \partial U} = 0, \quad t \geq \tau; \quad u(x,\tau) = u_\tau(x), \quad x \in U, \end{cases} \tag{1.2}$$

where $b > 0$, $u = u(x, t)$ is a real-valued function on $U \times [\tau, +\infty)$, $\tau \in \mathbb{R}$, U is an open bounded set of $\mathbb{R}^3$ with a smooth boundary ∂U, $h \in H_0^1(U) \cap H^2(U)$, $g(x, \cdot) \in C_b(\mathbb{R}, L^2(U))$ with

$$||g||^2 = \sup_{t \in \mathbb{R}} ||g(\cdot, t)||^2 < \infty; \tag{1.3}$$

f is a 3rd degree polynomial with a positive leading coefficient:

$$f(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3, \quad a_3 > 0, a_i \in \mathbb{R}, i = 0, 1, 2, 3; \tag{1.4}$$

$W(t)$ is a one-dimensional two-sided Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, the Borel σ-algebra $\mathcal{F}$ on Ω is generated by the compact open topology and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$.

When g is independent of t, the stochastic system (1.1) with multiplicative noise is autonomous. Caraballo, Fan, Li, Wang, Zhao et al. have studied the existence of its random attractor [17–23]; of those, for the case of $a_0 = a_2 = g \equiv 0$, Caraballo, Fan et al. obtained an upper bound of Hausdorff dimension of the random attractor [21–23].

For the stochastic reaction–diffusion Eq. (1.2) driven by the additive white noise on bounded and unbounded domains, the existence and boundedness of Hausdorff dimension of its random attractor have been studied by many authors, see [1–5,13–16,24–29] and other references.

Notice that Eq. (1.1) (or (1.2)) is a non-autonomous stochastic equation with a time-dependent external term g. For such a non-autonomous stochastic system, Wang established an efficacious theory about the existence and upper semi-continuity of the random attractor by introducing two parametric spaces [11–14].

Here, we focus on considering the system (1.1). First we prove that system (1.1) has a random attractor in $L^2(U)$ which is bounded in $H_0^1(U)$, then we apply our criteria to obtain an upper bound of fractal dimension of the random attractor when the coefficient b of the random term is suitable small, which implies that the random attractor of (1.1) can be embedded in a compact set of a finite-dimensional Euclidean space.

This paper is organized as follows. In Section 2, we give some sufficient conditions to obtain an upper bound of fractal dimension of a random invariant set for a non-autonomous random dynamical system. In Section 3, we apply our method to prove the boundedness of fractal dimension of the random attractor for the system (1.1). In Section 4, we present the boundedness of fractal dimension of the random attractor for the system (1.2). In Section 5, we consider a special case of (1.1) and (1.2) with $f(u) = u^3 - \beta u$, and obtain an upper bound of the fractal dimension for the random attractor. In Section 6, we present a conclusion of this article.

2. Fractal dimension of a random invariant set

In this section, we give some sufficient conditions to bound the fractal dimension of a random invariant set for a non-autonomous random dynamical system basing on the idea of [12,25]. Throughout this article, for simplicity, we identify "*a.e.* $\omega \in \Omega$" with "$\omega \in \Omega$".

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ be an ergodic metric dynamical system [40] and X be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$. Let a mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \to X$ be a continuous non-autonomous random dynamical system (RDS) (or called cocycle) on X over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$, (i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$ -measurable; (ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity on X; (iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau + s, \theta_s \omega, \Phi(s, \tau, \omega, \cdot))$; (iv) $\Phi(t, \tau, \omega, \cdot)$: $X \to X$ is continuous [12]. For the finiteness of fractal dimension of an invariant random set for Φ, we have the following result.

Assume that there exist a family of bounded closed random subsets $\{\chi(\tau, \omega)\}_{\tau \in \mathbb{R}, \omega \in \Omega}$ of X satisfying the following conditions: for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

(H1) there exists a tempered random variable R_ω such that the diameter $||\chi(\tau, \omega)||_X$ of $\chi(\tau, \omega)$ is bounded by R_ω, i.e., $\sup_{\tau \in \mathbb{R}} \sup_{u \in \chi(\tau,\omega)} ||u||_X \leq R_\omega < \infty$, and $R_{\theta_t \omega}$ is continuous in t for all $t \in \mathbb{R}$;