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Semi-discrete a priori error analysis for the optimal control of the unsteady Navier–Stokes equations with variational multiscale stabilization

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ABSTRACT

In this work, the optimal control problems of the unsteady Navier–Stokes equations with variational multiscale stabilization (VMS) are considered. At first, the first order continuous optimality conditions are obtained. Since the adjoint equation of the Navier–Stokes problem is a convection diffusion type system, then the same stabilization is applied to it. Semi discrete a priori error estimates are obtained for the state, adjoint state and control variables. Crank–Nicholson time discretization is used to get the fully discrete scheme. Numerical examples verify the theoretical findings and show the efficiency of the stabilization for higher Reynolds number.

1. Introduction

Optimal control theory of viscous flows has several applications in engineering science. There have been an interest in control problems of viscous problems in recent decades. Much of the work covers the existence and uniqueness results of the optimal control problems. Various numerical methods have been devoted to the solutions of the optimal control problems. The numerical methods are based on the computation of the derivatives of the function to be minimized. A gradient descent type method is frequently used to solve the control problems numerically.

In this paper, the optimal control problem of the Navier–Stokes equations are studied. Here, the function based approach *optimize-then-discretize* is used to get the optimality conditions. So that, the optimality system consists of the state and adjoint equations as coupled by an algebraic equation. In [1,8,14,15,19,30,31,33,35], optimal control of Navier–Stokes equations were studied. In literature, papers covering the optimal control of the Navier–Stokes equations are mostly related to the optimality conditions and numerical methods. Especially, second order numerical methods, sequential quadratic programming and semi-smooth Newton methods, have been analyzed. There are not much work focusing on the error analysis [14]. The originality of this paper comes from the application of the variational multiscale methods and a priori error analysis.

The Navier–Stokes equations provide mathematical models to describe the behavior of fluid flows. Here, small viscosity ν is considered. But, standard Galerkin finite element method fails to obtain the accurate solution due to the richness of flow scales. So that, a numerical stabilization should be used. Most popular stabilization methods for flow problems are residual based techniques as streamline upwind Galerkin (SUPG) and pressure stabilization methods, large eddy simulation (LES) methods and variational multiscale methods (VMS). In some techniques, numerical viscosity is added on all scales and this gives rise to some

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problems due to the richness of flow scales. Classical LES techniques attempt to model only the character of large scales. Then, various drawbacks like definition of appropriate boundary conditions for large scales and computations errors are encountered. Among other stabilization mechanisms, VMS is easy to carry out the analysis and implement to algorithm. The finite element method is chosen to solve this system in this study as a numerical method. In [25], the method used here was applied on a Navier–Stokes system.

A comparison of various stabilization techniques applied on an Oseen problem was studied in [7]. A symmetric stabilization technique, building the adjoint commute, and a quasi-optimal a priori estimate in the context of optimal control problems for the Oseen system was covered in [6]. In [2], the effect of the Galerkin/least-squares (GLS) stabilization on the finite element discretization of optimal control problems governed by the linear Oseen equations was studied. A convection-diffusion system was given in [10] with analysis of some well-known stabilization techniques. In [27], a discontinuous Galerkin finite element method (DG) with interior penalties for the optimal control problem of the convection-diffusion equation was studied and in [20] an edge stabilized Galerkin finite element method for the same optimal control system was considered. Moreover, local error estimates for SUPG solutions of advection-dominated elliptic linear-quadratic optimal control problems was studied in [18]. Similarly, the local (DG) for optimal control problem governed by convection-diffusion equations was analyzed in [36]. In [14], the authors considered the optimal control problem of Navier–Stokes equations without any stabilization. In [9], a subgrid-scale stabilization scheme, similar to the idea used in this study was cast to an optimal control problem of an Oseen system and error bounds were obtained.

In this study, a projection-based VMS approach [9,12,23] is applied to both the state equation of a time dependent optimal control problem, Navier–Stokes, and its adjoint equation. In this technique, we add the global stabilization first and then subtract its effect from the large scales which are defined through projections. Thus, stabilization acts only on the smallest resolved scales for both state and adjoint equations [12].

The organization of the paper is as follows: Firstly, optimal control problem is defined and some notational notes and mathematical preliminaries are given. Then, the finite element discretization of the optimal control problem is considered with VMS. A priori error analysis of the overall system is given next. Then, we conclude our study with numerical examples to verify the effectiveness of the method.

2. Problem formulation

We define Ω be a bounded polygonal domain in $\mathbb{R}^2$, and its Lipschitz boundary be $\Sigma = \partial\Omega$. Let be $Q = [0, T] \times \Omega$, where $T > 0$. We consider the following optimal control problem of the unsteady Navier–Stokes equations:

$$\min_{(y,u)} J(y, u) = \frac{1}{2}\int_Q ((y(t, x) - y_d(t, x))^2 + \alpha(u(t, x))^2)dxdt \tag{1}$$

$$\begin{aligned} \text{subject to } \; y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p &= f + u \;\text{ in } Q,\\ \nabla\cdot y &= 0 \;\text{ in } Q,\\ y &= 0 \;\text{ on } [0, T]\times\Sigma,\\ y(0, x) &= y_0(x) \;\text{ in } \Omega, \end{aligned} \tag{2}$$

where the state variable $y : Q \mapsto \mathbb{R}^2$ is the fluid velocity, $p : Q \mapsto \mathbb{R}$ denotes the fluid pressure and u is the control variable. The kinematic viscosity, which is the multiplicative reciprocal of the Reynolds number, is denoted by $\nu > 0$. Here, $\alpha > 0$ stands for the regularization parameter, $f(t, x)$ is a given function and y_d is the desired state.

Norms and inner products for Sobolev and Lebesgue spaces are used as in [3]. Let us consider the following notations:

$$L^r(0, T; X) = \left\{z : [0, T] \to X \text{ measurable: } \int_0^T \|z(t)\|_X^r dt < \infty\right\}$$

with the norm

$$\|z(t)\|_{L^r(0,T;X)} = \begin{cases} \left(\int_0^T \|z(t)\|_X^r dt\right)^{1/r}, & \text{if } \; 1 \le t < \infty\\ \operatorname*{ess\,sup}_{t\in(0,T)} \|z(t)\|_X, & \text{if } \; r = \infty. \end{cases}$$

The spatial parts of the velocity, pressure and control spaces are denoted by $Y = H_0^1(\Omega)$, $M = L_0^2(\Omega)$ and $U = L^2(\Omega)$, respectively. The dual space of $Y = H_0^1(\Omega)$, namely the space $H^{-1}(\Omega)$ is equipped with the -1-norm

$$\|t\|_{-1} = \sup_{v\in Y} \frac{\langle t, v\rangle}{\|v\|_1}. \tag{3}$$

Here, $\langle\cdot, \cdot\rangle$ denotes the duality pairing. For simplicity, we let $\|v\|^2$ and $\|v\|_1^2$ denote the norms $\|v\|^2_{L^2(\Omega)} = \int_\Omega v\cdot v\,dx$ and $\|v\|^2_{H^1(\Omega)} = \int_\Omega (v\cdot v + \nabla v\cdot\nabla v)\,dx$, respectively.

We shall give a summary of the mathematical theory for the unsteady Navier–Stokes equations:

$$y_t - \nu\Delta y + (y\cdot\nabla)y + \nabla p = f \;\text{ in } Q,$$