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A numerical verification method for nonlinear functional equations based on infinite-dimensional Newton-like iteration

Yoshitaka Watanabe[a,b,*], Mitsuhiro T. Nakao[c]

[a] Research Institute for Information Technology, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8518, Japan
[b] CREST, Japan Science and Technology Agency, Japan
[c] National Institute of Technology, Sasebo College, 1-1 Okishin-cho, Sasebo, Nagasaki 857-1193, Japan

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ABSTRACT

This paper describes a numerical verification of solutions for infinite-dimensional functional equations based on residual forms and Newton-like iteration. The method is based upon a verification method previously developed by the authors. Several computer-assisted proofs for differential equations, including nonlinear partial differential equations, are presented.

1. Introduction

Let $\hat{X}$ be a Banach space and X, Y be Hilbert spaces satisfying $\hat{X} \hookrightarrow X \hookrightarrow Y$ with the compactness of the embedding $\hat{X} \hookrightarrow X$. Let $\mathcal{A}$ be a linear operator from $\hat{X}$ to Y, and f: $X \to Y$ be a continuous operator which maps bounded sets in X into bounded sets in Y. Consider the problem of finding u which satisfies

$$\mathcal{A}u = f(u). \tag{1}$$

For nonlinear differential equations, in Eq. (1), the operator $\mathcal{A}$ includes both the highest order differential term and f, which is the only other term and is nonlinear (see Section 5).

This paper presents a numerical verification algorithm IN-Linz for Eq. (1) by using a residual form and Newton-like iteration. Here "numerical verification" means a computer-assisted numerical method for proving the existence of a solution in an explicit neighborhood of an approximate solution. IN-Linz comes from "Infinite", "Newton", and "Linearized". Our proposed approach is in fact an extension of the methods presented in [12,25]. In [12], the authors considered some numerical verification methods for second-order semilinear elliptic boundary value problems. Our proposed approach is based on some infinite-dimensional fixed-point theorems using a Newton-like operator and a projection into a finite-dimensional subspace, and constructive error estimates of the projection play an important and essential role. Specifically, our proposed method is a generalization of a method described in [12] which can be applied to nonlinear functional equations in Hilbert spaces. In [25] one of the authors proposed a numerical verification method based on sequential iteration. Although this method can be applied to general nonlinear functional equations in Banach spaces, this requires that the formulated compact map

* Corresponding author at: Research Institute for Information Technology, Kyushu University, 6-10-1 Hakozaki, Higashi-ku, Fukuoka 812-8518, Japan. Tel.: +81 92 642 2295.

E-mail addresses: watanabe@cc.kyushu-u.ac.jp (Y. Watanabe), mtnakao@sasebo.ac.jp (M.T. Nakao).

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be retractive in some neighborhood of the fixed-point to be verified. IN-Linz is an extension of the sequential iteration of [25] to a Newton-like iteration. We consider a linearized operator (denoted by $\mathscr{L}$) of (1), verify the invertibility of $\mathscr{L}$ and compute guaranteed norm bounds for $\mathscr{L}^{-1}$ by applying the same principle as in [26]. After that, we confirm the existence of solutions by proving the contractility of the infinite-dimensional Newton-like operator with a residual form.

We now give some additional brief remarks on another Newton-type verification methods based on bounds for $\mathscr{L}^{-1}$ proposed by Plum [15–17], Oishi [13] and Takayasu et al. [14], for example. From the standpoint based upon the infinite-dimensional Newton-type method of the residual type, IN-Linz is close to Plum's and Oishi's method. The method described by Plum [15–17] is based on eigenvalue bounds, which are obtained by the Rayleigh–Ritz and the Lehmann–Goerisch method with additional base functions and some homotopic steps, together with verified computations for rather small matrix eigenvalue problems. It does not need any infinite-dimensional projection error estimates and is applicable to differential equation problems on unbounded domains. However, when $\mathscr{L}$ is non-self-adjoint, higher-order base functions are needed since then eigenvalue bounds for $\mathscr{L}^*\mathscr{L}$ ($\mathscr{L}^*$ is an adjoint operator) are required. In contrast, the verified computation in the present paper does not need higher-order finite dimensional spaces because it is based on the weak formulation.

Oishi's method bounds the operator norm of $\mathscr{L}^{-1}$ by estimations based on numerical computation of the matrix norm of its Galerkin approximation, together with error bounds for the Galerkin projection. Although this procedure, in principle, could be applicable to general Banach spaces and operators, application results in [13,14] are only ordinary differential equations. The present paper reports on some verification results for partial differential equations, and it should be emphasized that norm bounds of $\mathscr{L}^{-1}$ by Theorem 3 in Section 4 are expected to converge, as the Galerkin space increases, to the exact operator norm [26] and to provide accurate and efficient enclosure results for the solution of nonlinear problems.

The remainder of the paper is organized as follows. Section 2 describes residual forms by using approximate solutions. Section 3 concerns a verification algorithm IN-Linz which is based on infinite-dimensional Newton-type iterations. In Section 4, we introduce a method to compute an upper bound on the norm of $\mathscr{L}^{-1}$, as well as the invertibility of $\mathscr{L}$. The paper concludes with some computer-assisted results.

2. Residual forms

This section introduces two kind of residual forms of Eq. (1) by using approximate solutions. Define the inner products and norms of the Hilbert spaces X and Y by $(u, v)_X$, $(u, v)_Y$ and $\|u\|_X = \sqrt{(u,u)_X}$, $\|u\|_Y = \sqrt{(u,u)_Y}$, respectively. Suppose that the operator $\mathcal{A}$ has the following properties:

A1. For each $\phi \in Y$, $\mathcal{A}\psi = \phi$ has the unique solution $\psi \in \hat{X}$ and this mapping $\phi \mapsto \psi$ is continuous.
A2. It holds that

$$(u, v)_X = (\mathcal{A}u, v)_Y, \quad \forall u \in \hat{X}, \quad \forall v \in X. \tag{2}$$

Let $u_h \in X$ be an approximate solution for problem (1). Using $u_h \in X$, we consider two types of residual equations.

2.1. Direct residual form

When $u_h \in X$ satisfies $\mathcal{A}u_h \in Y$, Eq. (1) can be rewritten as

$$\mathcal{A}w = f(w + u_h) - \mathcal{A}u_h \tag{3}$$

to find the residual

$$w := u - u_h.$$

Denoting $g(w) := f(w + u_h) - \mathcal{A}u_h : X \to Y$, we obtain the residual equation $\mathcal{A}w = g(w)$. It is expected that if u_h is an accurate approximation of u for Eq. (1), the residual term g will be small.

2.2. X*-type residual form

In the case of $\mathcal{A}u_h$ not belonging to Y, we apply the following "X^*-type" residual formulation [6]. "X^*" here means the dual space of X. Let X_h be a finite-dimensional approximation subspace of X dependent on the parameter $h > 0$. For example, X_h is taken to be a finite element subspace with mesh size h. Define the orthogonal projection P_h: $X \to X_h$ by

$$(v - P_h v, v_h)_X = 0, \quad \forall v_h \in X_h, \tag{4}$$

and suppose that P_h has the following approximation property.

A3. There exists $C(h) > 0$ such that

$$\|(I - P_h)u\|_X \le C(h)\|\mathcal{A}u\|_Y, \quad \forall u \in \hat{X}. \tag{5}$$