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Quadratic spline collocation method for the time fractional subdiffusion equation

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ABSTRACT

In this paper, exploiting the quadratic spline collocation (QSC) method, we numerically solve the time fractional subdiffusion equation with Dirichelt boundary value conditions. The coefficient matrix of the discretized linear system is investigated in detail. Theoretical analyses and numerical examples demonstrate the proposed technique can enjoy the global error bound with $O(\tau^3 + h^3)$ under the L_∞ norm provided that the solution $v(x,t)$ has four-order continual derivative with respects to x and t, and it can achieve the accuracy of $O(\tau^4 + h^4)$ at collocation points, where τ, h are the step sizes in time and space, respectively.

1. Introduction

In the continuous time random walk (CTRW) theory, combining with some initial and boundary conditions, some researchers have presented the governing mathematical model

$$ {}_C\mathbf{D}_{0,t}^{\beta} u = k \frac{\partial^{\alpha} u}{\partial |x|^{\alpha}} + f(x,t) \tag{1} $$

to describe the diffusion process in different mediums [1,2], where ${}_C\mathbf{D}_{0,t}^{\beta} u$ and $\frac{\partial^{2\alpha} u}{\partial |x|^{2\alpha}}$ are respectively the Caputo and Riesz fractional derivatives [3].

When $\alpha = 2, \beta = 1$, Eq. (1) is the traditional integer-order differential model problem. As we know, in a highly nonhomogeneous medium, the corresponding probability density of the concentration field obtained by traditional model may have a heavier tail than the Gaussian density [1,4]. Nevertheless, it finds that when one takes the condition $\frac{2\beta}{\alpha} < 1$, which corresponds to this case called the subdiffusion motion, may be more adequate to describe this phenomenon, refer to [5,6] for details.

Due to the nonlocal property of fractional derivatives, it is usually difficult to obtain the analytical solutions [3]. As an alternative, more and more researchers have began to increasingly focus on the efficient numerical solutions of fractional differential equations.

For the cases of space and space-time fractional subdiffusion models, many prominent numerical approaches have recently been presented, including the finite volume method [5], the fast semi-implicit difference method [7], the implicit Euler scheme [8], the alternating direct method [9], and other numerical methods (see e.g. [10,11]).

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In this paper, we are interested in the following time-fractional subdiffusion equation

$$\begin{cases} {}_C\mathbf{D}_{0,t}^{\beta} u = k\dfrac{\partial^2 u}{\partial x^2} + f(x,t),\ (x,t) \in I \times (0,T], \quad I = (a,b), T > 0; \\ u(x,0) = \phi(x), x \in I; \\ u(a,t) = \varphi(t), u(b,t) = \psi(t), t \in (0,T], \end{cases} \tag{2}$$

where $0 < \beta < 1$, and ${}_C\mathbf{D}_{0,t}^{\beta}$ is the βth-order Caputo derivative operator of the form

$${}_C\mathbf{D}_{0,t}^{\beta} u = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} \frac{\partial u}{\partial s} ds, \tag{3}$$

$k > 0$ is the diffusion constant, $\Gamma(\cdot)$ is the gamma function, $\phi(x)$, $\varphi(t)$, $\psi(t)$, $f(x,t)$ are all specified smooth functions, and $u(x,t)$ is the unknown function to be solved.

For the numerical solution of (2), one papular idea in the existed work is employing the relation between the Caputo derivative and the Riemann-Liouville derivative [3] to transform it into the following equivalent system

$$\begin{cases} \dfrac{\partial u}{\partial t} = {}_0\mathbf{D}_t^{1-\beta}\left[k\dfrac{\partial^2 u}{\partial x^2}\right] + g(x,t),\ (x,t) \in I \times (0,T]; \\ u(x,0) = \phi(x), x \in I; \\ u(a,t) = \varphi(t), u(b,t) = \psi(t), t \in (0,T], \end{cases} \tag{4}$$

where $g(x,t) = {}_0\mathbf{D}_t^{1-\beta} f(x,t)$, ${}_0\mathbf{D}_t^{\beta} s(x,t)$ is the Riemann–Liouville derivative of function $s(x,t)$[3], and then designing some schemes to discretize Eq. (4)[12–15].

There are also some approaches which were developed to directly solve Eq. (2). Gao and Sun [16] introduced a compact finite difference scheme with $O(\tau^{2-\beta} + h^4)$ order accuracy. Jiang and Ma [17] presented an $O(\tau^{2-\beta} + h^r)$ order finite element method with r the degree of the polynomial test function space. Based on piecewise linear functions, Jin et al. [18] established a lumped mass Galerkin FEM. Azizi [19], Chen [20], and Lin [21] respectively introduced some spectral methods. Using FEM in space, and fractional linear multistep method (FLMM) in time, Zeng et al. [22,23] successively proposed two discretized approaches with accuracy of $O(\tau^{2-\beta} + h^{r+1})$ and $O(\tau^2 + h^{r+1})$. For $\beta \in (0, 0.9569347]$, Cui and Sun [24] derived a compact finite difference scheme with $O(\tau^3 + h^4)$ order accuracy. Recently, employing the Jacobi polynomials and Fourier-like basis functions, Zheng [4] also proposed a high order space-time spectral method for a time fractional Fokker–Planck equation which is a generalization of subdiffusion Eq. (2), proved theoretically an exponential convergence may be achieved when the exact solution is sufficiently smooth. Due to the lack of orthogonality for the employed basis functions, this spectral discretization results in a full stiffness matrix, which requires an efficient solver and large computational memory for handling the resulting linear system. Additionally, some numerical schemes have been introduced for high-dimensional and/or variable order models, refer to [25–27] for details.

From a lot of early established numerical methods, it notes that the resulted accuracy seems to be considerably susceptible to the order β. The contribution of this paper is to establish a novel collocation method for (2) via taking quadratic spline polynomials as basic functions. The key idea of the technique is that, by the initial value in time, we transform (2) into an equivalent system, where the new unknown function $v(x,t) = \frac{\partial u}{\partial t}$, and then, use two interpolation operators successively to approximate $v(x,t)$ and $\frac{\partial^2 v}{\partial x^2}$. Later, in the theoretical analyses and numerical examples, it can find the accuracy of the proposed collocation method is independent of β.

The outline of this paper is as follows. In Section 2, some preliminaries are provided, which are useful to construct the QSC method. In Section 3, based on the quadratic spline function, the QSC method is constructed, the corresponding collocation equations and the coefficient matrix are also given. Meanwhile, the existence, uniqueness, convergence and stability of the proposed numerical scheme are studied. In Section 4, by comparing with another recently presented scheme, some numerical examples are given to illustrate the effectiveness of the proposed technique. Finally, some conclusions about the established method are drawn.

2. Preliminaries

Define respectively $\rho_h = \{x_i\}_{i=1}^{N_h+1}$ and $\rho_t = \{t_i\}_{i=1}^{N_t+1}$ as uniform partitions of the interval $\bar{I} = [a,b]$ and $[0,T]$ with

$$x_i = a + (i-1)h, t_j = (j-1)\tau, \quad i = 1, \ldots, \quad N_h + 1, j = 1, \ldots, \quad N_t + 1, \quad h = \frac{b-a}{N_h}, \quad \tau = \frac{T}{N_t}.$$

Let

$$\eta_i = \frac{x_i + x_{i-1}}{2}, \quad i = 2, 3, \ldots, \quad N_h + 1, \tau_j = \frac{t_j + t_{j-1}}{2}, \quad j = 2, 3, \ldots, N_t + 1,$$

then the collocation points in $(a,b) \times (0,T)$ can be defined by $\{(\eta_i, \tau_j)\}, i = 2, 3, \ldots, N_h + 1, j = 2, 3, \ldots, N_t + 1$, the center of each gridding cell. Moreover, with parameter $\theta \in (0, \frac{1}{2})$, we take

$$\eta_1 = a, \quad \eta_{N_h+2} = b, \quad \tau_1 = \theta\tau, \quad \tau_{N_t+2} = T$$