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Semilocal and local convergence of a fifth order iteration with Fréchet derivative satisfying Hölder condition

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ABSTRACT

The semilocal and local convergence in Banach spaces is described for a fifth order iteration for the solutions of nonlinear equations when the Fréchet derivative satisfies the Hölder condition. The Hölder condition generalizes the Lipschitz condition. The importance of our work lies in the fact that many examples are available which fail to satisfy the Lipschitz condition but satisfy the Hölder condition. The existence and uniqueness theorem is established with error bounds for the solution. The convergence analysis is finally worked out on different examples and convergence balls for each of them are obtained. These examples include nonlinear Hammerstein and Fredholm integral equations and a boundary value problem. It is found that the larger radius of convergence balls is obtained for all the examples in comparison to existing methods using stronger conditions.

1. Introduction

Nonlinear equations and their methods for solutions in Banach space setting are extensively studied problems in numerical analysis and scientific computing. It is found that many real life problems arising in science and engineering [3,4] are reduced to find solutions of such equations. It is because their mathematical models involve scalar equations, system of equations, differential equations, integral equations, etc. whose solutions require solving thousands of such equations. The dynamical systems represented by differential equations also lead to solve these equations. Generally, iterative methods with convergence analysis including semilocal and local are used for solving these equations. The semilocal convergence [7,9] uses information given at initial point whereas local convergence [6,8,10] uses information around the solution. Another important problem which is to be considered for these iterative methods is the domains of convergence balls. In general, the convergence domain of an iterative method is small and one always tries to enlarge it by considering additional hypothesis.

Let S and T are Banach spaces. Consider solving nonlinear equations

$$G(x) = 0 \tag{1.1}$$

where, $G: D \subseteq S \rightarrow T$ be twice Fréchet differentiable in open convex region $D_0 \subseteq D$ with values in T. Here, we consider the semilocal and local convergence of iteration [1] given by

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$$
\begin{aligned}
y_n &= x_n - \Gamma_n G(x_n), \\
z_n &= y_n - 5\Gamma_n G(y_n), \\
x_{n+1} &= z_n - \frac{1}{5}\Gamma_n[-16G(y_n) + G(z_n)], n \geq 0
\end{aligned} \tag{1.2}
$$

where, $\Gamma_n = G'(x_n)^{-1}$ and x_0 is the starting point. Let G' satisfies the Hölder condition given by $\|G'(x) - G'(y)\| \leq K\|x - y\|^q, x, y \in D, q \in (0, 1]$. It is worth mentioning that higher order convergence requires computation of derivatives of higher order which are very expansive in general. For example, the third order Chebyshev–Halley type methods [16] require evaluation of G'' which either does not exist or computationally difficult to evaluate. But higher order methods have their importance as in some applications involving stiff system of equations require faster convergence. Also, it is found that many integral equations involve G'' which is inexpensive and diagonal by blocks [12]. The semilocal convergence using recurrence relations in [19] and the local convergence in [6] are discussed for a family of iterative methods of order three under Lipschitz condition on G'. Argyros et al. [13] considered parametric Chebyshev–Halley-type methods with multipoints having high order of convergence with G' satisfying Lipschitz condition for their local convergence analysis. The convergence of a modified Halley-like method of high convergence order uses Lipschtiz condition in [14]. A study on the local convergence analysis and the dynamics of Chebyshev–Halley-type methods of convergence order at least five free from second derivative under Lipschitz condition is studied in [15]. The semilocal convergence using recurrence relations in [2] and the local convergence in [5] are studied for (1.2) under Lipschitz condition on G'. It is to be noted that the Lipschitz condition is a particular case of Hölder condition. Moreover, examples for which G' satisfy Hölder condition but fail to satisfy Lipschitz condition can be constructed.

Example 1.1. Consider an integral equation given by

$$G(x)(s) = x(s) - 1 - 3\int_0^1 G_1(s, u)x(u)^{5/4}du,$$

with $x(s) \in \mathcal{C}[0, 1]$ and $G_1(s, u)$ denotes the Green function.

Therefore,

$$\|G'(x) - G'(y)\| \leq \frac{15}{32}\|x - y\|^{1/4}$$

where, $\|G_1(s, u)\| = 1/8$. Clearly G' satisfies Hölder condition for $q = \frac{1}{4}$ whereas Lipschitz condition fails.

Example 1.2. Consider the Fredholm integral equation

$$G(x)(s) = x(s) - f(s) - \lambda\int_0^1 \frac{s}{s+u}x(u)^{1+q}du,$$

with $s \in [0, 1], q \in (0, 1], x, f \in C[0, 1]$ and λ is a real number.

Therefore,

$$\|G'(x) - G'(y)\| \leq |\lambda|(1+q)\log 2\|x - y\|^q, \quad x, y \in D.$$

Clearly, Lipschitz condition fails and Hölder condition holds on G' for $q \in (0, 1)$.

Using recurrence relations, Ezquerro et al. [17] and Parida and Gupta [18] developed the semilocal convergence analysis of Newton-like methods of order three with G' and G'' satisfying the Hölder condition. Recently, Argyros and George [11] established the convergence of deformed Halley method of third order locally under Hölder condition on G'. It is defined by

$$
\begin{aligned}
y_n &= x_n - \Gamma_n G(x_n), \\
z_n &= x_n + \alpha\Gamma_n G(x_n), \\
L_n &= \frac{1}{\lambda}\Gamma_n[G'(x_n + \lambda(z_n - x_n)) - G'(x_n)], \\
x_{n+1} &= y_n + \frac{1}{2}L_n\left(I - \frac{1}{2}L_n\right)^{-1}(y_n - x_n), n \geq 0
\end{aligned} \tag{1.3}
$$

where, $\Gamma_n = G'(x_n)^{-1}$, $\lambda \in (0, 1]$, $\alpha \in \mathbb{R}$ and x_0 is the starting point. This method does not require the computation of expensive G''.

In this paper, the semilocal and local convergence in Banach spaces is described for a fifth order iteration for the solutions of nonlinear equations when the Fréchet derivative satisfies the Hölder condition. The Hölder condition generalizes the Lipschtiz condition. The importance of our work lies in the fact that many examples are available which fail to satisfy the Lipschtiz condition but satisfy the Hölder condition. The existence and uniqueness theorem is established with error bounds for the solution. The convergence analysis is finally worked out on different examples and convergence balls for each of them are obtained. These examples include nonlinear Hammerstein and Fredholm integral equations and a boundary value problem. It is found that the larger radius of convergence balls is obtained for all the examples in comparison to existing methods using stronger conditions.

This paper is arranged in the following manner. Introduction forms Section 1. The semilocal convergence of an iteration of fifth order under Hölder condition on G' is established in Banach spaces in Section 2. A theorem for the existence and uniqueness