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A combined ternary 4-point subdivision scheme

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ABSTRACT

A new combined ternary 4-point subdivision scheme is introduced that generates interpolating C^1 limiting curves and approximating C^1, C^2, C^3 limiting curves. Laurent polynomial method is used to investigate the derivative continuity of the proposed subdivision scheme. The role of the tension parameters in the scheme are demonstrated using a few examples.

1. Introduction

Subdivision is a simple and popular method to generate smooth limiting curves from discrete set of data points. Subdivision schemes are widely used in many fields like computer graphics, computer aided geometric design and image processing etc. The popular interpolating subdivision schemes are binary 4-point interpolating subdivision scheme introduced by Dubuc [2] and independently Dyn et al. [3] with tension parameter. It generates the limiting curves of C^1 continuous. Tan et al. [9] introduced a 4-point subdivision scheme which generates the limiting curve of C^3 continuous.

Hassan et al. [4] proposed a ternary 4-point interpolating subdivision scheme that generates C^2 limiting curves for certain range of tension parameter. Siddiqi and Rehan [7] also developed a ternary 4-point interpolating subdivision scheme that generates the limiting curve of C^2 continuity. Pan et al. [10] introduced a new combined approximating and interpolating subdivision scheme which generates the limiting curve of C^2 continuous. Hormann and Sabin [5] constructed the subdivision schemes according to the properties such as support, precision set and so on. Augsdörfer et al. [6] took a geometric approach to generate subdivision curve for achieving different requirements. Siddiqi and Rehan [8] introduced a new ternary $2N$-point Lagrange subdivision scheme that generates the limiting curves of C^3, C^4 and C^5 continuities for $N = 2, 3$ and 4, respectively.

A step of subdivision can be taken as a highly simple step. By developing this step of the subdivision, we can generates families of limiting curves. A new combined ternary 4-point subdivision scheme is proposed using tension parameters in the mask f_{3i}^{k+1}, f_{3i+1}^{k+1} and f_{3i+2}^{k+1}. These tension parameters give flexibility to generate families of approximating and interpolating limiting curves. A new combined ternary 4-point approximating and interpolating subdivision scheme is defined as

$$f_{3i}^{k+1} = a_0 f_{i-2}^k + a_1 f_{i-1}^k + a_2 f_i^k + a_3 f_{i+1}^k + a_4 f_{i+2}^k,$$

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$$f_{3i+1}^{k+1} = b_0 f_{i-1}^k + b_1 f_i^k + b_2 f_{i+1}^k + b_3 f_{i+2}^k,$$
$$f_{3i+1}^{k+1} = b_3 f_{i-1}^k + b_2 f_i^k + b_1 f_{i+1}^k + b_0 f_{i+2}^k, \tag{1.1}$$

where

$$a_0 = a_4 = -\alpha, \quad a_1 = a_3 = 4\alpha, \quad a_2 = (1-6\alpha),$$
$$b_0 = \left(\frac{-5}{81} - (1-\alpha)\beta\right), \quad b_1 = \left(\frac{20}{27} + (1-\alpha)\beta\right), \quad b_2 = \left(\frac{10}{27} + (1-\alpha)\beta\right), \quad b_3 = \left(\frac{-4}{81} - (1-\alpha)\beta\right).$$

In the proposed scheme α and β are shape control parameters. For $\alpha = 0$ and $\beta = 0$, the proposed subdivision scheme (1.1) is the ternary 4-point Deslauriers and Dubuc [1] subdivision scheme that generates an interpolating C^1 limiting curve. For $\beta = 0$, the proposed subdivision scheme (1.1) rebuilds the mask of the scheme introduced by Siddiqi and Rehan [8].

2. Basic notion

A subdivision scheme with the initial values $f^0 = \{f_j^0 \in R : j \in Z\}$ defines recursively new discrete values $f^k = \{f_j^k \in R : j \in Z\}$ on finer levels by linear sums of existing values as follows

$$f_i^{k+1} = \sum_{j\in Z} a_{i-3j} f_j^k, \quad k \in Z_+,$$

where the sequence $a = \{a_j : j \in Z\}$ is termed the mask of the given subdivision scheme. Here, S denote the rule at each step and have the relation

$$f^k = S^k f^0. \tag{2.1}$$

A subdivision scheme S is said to be C^m if for the initial data $f^0 = \{\delta_{j,0} : j \in Z\}$, there exist a limit function $f = S^\infty f^0 \in C^m(R)$, $f \neq 0$, satisfying

$$\lim_{k\to\infty} \sup_{j\in Z} |f_j^k - f(3^{-k} j)| = 0.$$

For the convergent subdivision scheme S, the corresponding mask $\{a_j\}, j \in Z$ necessarily satisfies

$$\sum_{j\in Z} a_{3j} = \sum_{j\in Z} a_{3j+1} = \sum_{j\in Z} a_{3j+2} = 1.$$

Introducing a symbol called the Laurent polynomial $a(z) = \sum_{j\in Z} a_j z^j$ of a mask $\{a_j\}, j \in Z$ with finite support. The corresponding symbols play an efficient role to analyze the convergence and smoothness of subdivision scheme. Define the Laurent polynomial $a^{[k]}(z)$, $k \in N$, by

$$a^{[k]}(z) = a(z)a(z^3), \ldots, a(z^{3^{k-1}}) = \sum_{j\in Z} a_j^{[k]} z^j. \tag{2.2}$$

Using the coefficients $a_j^{[k]}$ in Eq. (2.2), the norm of the iterative subdivision scheme S^k in Eq. (2.1) is defined as

$$\|S^k\|_\infty = max\left\{\sum_{j\in Z} |a_{\gamma+3^k j}^{[k]}| : \gamma = 0, 1, \ldots, 3^k - 1\right\}. \tag{2.3}$$

3. Smoothness analysis

Theorem 1. *A combined ternary 4-point subdivision scheme defined in Eq. (1.1) converges and generates, C^1 interpolating curve when $\alpha = 0$ and $-0.0555 < \beta < 0.11$, C^1 approximating curve when $-0.03 < \alpha < 0.075$ and $\beta = 0$, C^2 approximating curve when $0.004115 < \alpha < 0.018$ and $\beta = 0$, C^3 approximating curve when $0.016461 < \alpha < 0.01725$ and $\beta = 0$.*

Proof. The generating function corresponding to the proposed combined ternary 4-point subdivision scheme defined in Eq. (1.1) has the following sequence of coefficients

$$a = (a_i) = (\ldots, a_0, b_3, b_0, a_1, b_2, b_1, a_2, b_1, b_2, a_1, b_0, b_3, a_0, \ldots).$$

The Laurent polynomial $a(z)$ for the mask of the scheme can be written as

$$a(z) = a_0 z^{-6} + b_3 z^{-5} + b_0 z^{-4} + a_1 z^{-3} + b_2 z^{-2} + b_1 z^{-1} + a_2 z^0 + b_1 z^1 + b_2 z^2 + a_1 z^3 + b_0 z^4 + b_3 z^5 + a_0 z^6.$$

Laurent polynomial method is used to prove the smoothness of the scheme to be C^2 and C^3. Taking

$$a_m(z) = \left(\frac{3z^2}{1+z+z^2}\right) a_{m-1}(z) = \left(\frac{3z^2}{1+z+z^2}\right)^m a(z).$$