Applied Mathematics and Computation 276 (2016) 356–366

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Integrable solutions of a generalized mixed-type functional integral equation

Haydar Abdel Hamid*, Waad Al Sayed

Faculty of Sciences and Humanities, Fahad Bin Sultan University, P.O. Box 15700, Tabuk 71454, Saudi Arabia

ARTICLE INFO

MSC:
47G10
47H10
47H30

Keywords:
Krasnosel'skii type fixed point theorem
(ws)-compact operator
Measure of weak noncompactness
Separate contraction
Mixed-type nonlinear functional integral equation

ABSTRACT

In this work, we prove the existence of integrable solutions for the following generalized mixed-type nonlinear functional integral equation

$$x(t) = g(t, (Tx)(t)) + f\left(t, \int_0^t k(t,s)u(t,s,(Qx)(s))\, ds\right), \ t \in [0, \infty).$$

Our result is established by means of a Krasnosel'skii type fixed point theorem proved by Taoudi (2009). In the last section we give an example to illustrate our result.

1. Introduction

Consider the following mixed-type nonlinear functional integral equation

$$x(t) = g(t, (Tx)(t)) + f\left(t, \int_0^t k(t,s)u(t,s,(Qx)(s))\, ds\right), \ t \in [0, \infty). \tag{1.1}$$

where $f, g: \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $k: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$, $u: \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $(Tx)(t)$, $(Qx)(t)$ are given while $x(t)$ is an unknown function.

In [1], the authors studied the existence of integrable solutions of the following special case of the Eq. (1.1)

$$x(t) = f\left(t, \int_0^t k(t,s)u(s,x(s))\, ds\right), \ t \in [0, \infty).$$

The following generalization of this equation

$$x(t) = g(t, x(t)) + f\left(t, \int_0^t k(t,s)u(s,x(s))\, ds\right), \ t \in [0, \infty),$$

has been studied by Taoudi [2] with the presence of the perturbation term g.

In this paper, we are going to study the existence of integrable solutions of the more general form (1.1). A classical point of view for solving Eq. (1.1) is to write the equation in the form

* Corresponding author. Tel.: +966 549712688.
E-mail addresses: habdelhamid@fbsu.edu.sa (H. Abdel Hamid), wsaid@fbsu.edu.sa (W. Al Sayed).

http://dx.doi.org/10.1016/j.amc.2015.12.024
0096-3003/

$$Ax + Bx = x, \tag{1.2}$$

where A and B are two nonlinear operators.

Fixed point theory seems to be one of the most natural and powerful tools in studying the solvability of integral equations in the form (1.2). In [3], Krasnosel'skii established a fixed point theorem which was frequently used to solve some special integral equations in the form (1.2), see [4,5]. Krasnosel'skii combined the famous Banach contraction principle of Banach [6] and the classical Schauder fixed point theorem of Schauder [7] to prove that $A + B$ has a fixed point in a nonempty closed convex subset $\mathcal{M}$ of a real Banach space X if A and B satisfy the following conditions (see [3,8]):

- A is continuous and compact;
- B is a strict contraction;
- $A\mathcal{M} + B\mathcal{M} \subseteq \mathcal{M}$.

Generalizations and improvements of such a result have been made in several directions, we refer for example to the papers [9–19] and the references therein. A deep variant of Krasnosel'skii type fixed point theorems is established by Latrach and Taoudi in [14]. In [2], Taoudi established an improvement of this variant. The most important advantage of [14, Theorem 2.1] and [2, Theorem 3.7] is that the operator A is not assumed to be compact. Let us recall the Krasnosel'skii type fixed point theorem [2, Theorem 3.7].

Theorem 1.1. *Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A : \mathcal{M} \to X$ and $B : \mathcal{M} \to X$ such that:*

(i) *A is (ws)-compact;*
(ii) *There exists $\gamma \in [0, 1[$ such that $\mu(AS + BS) \leq \gamma\mu(S)$ for all $S \subseteq \mathcal{M}$; here μ is an arbitrary measure of weak noncompactness on X;*
(iii) *B is a separate contraction;*
(iv) *$A\mathcal{M} + B\mathcal{M} \subseteq \mathcal{M}$.*

Then there is $x \in \mathcal{M}$ such that $Ax + Bx = x$

Our aim is to prove the existence of solutions of Eq. (1.1) in the space $L^1(\mathbb{R}_+)$ of Lebesgue integrable functions on the the real half-axis $\mathbb{R}_+ = [0, \infty)$. Theorem 1.1 plays a crucial role in establishing our result in Theorem 3.1. A technique of measures of weak noncompactness used in [1] will be implemented. The result obtained in this paper generalizes the result of Taoudi [2] to a more general equation such as Eq. (1.1) and extends the technique used in [1] to our more general context under special assumptions.

The outline of this paper is as follows. In Section 2, we recall some notations, definitions and basic tools which will be used in our investigations. Section 3 is devoted to state our main result and to prove some preliminary results. In Section 4, we prove our main result. In the last section we construct a nontrivial example illustrating our result.

2. Preliminaries

In this section we recall without proofs some of useful facts on Lebesgue space $L^1(\mathbb{R}_+)$, the superposition operator, contractions, (*ws*)-compact operators and measures of weak noncompactness.

2.1. The Lebesgue space

Let $\mathbb{R}$ be the set of real numbers and let $\mathbb{R}_+$ be the interval $[0, +\infty)$. For a fixed Lebesgue measurable subset I of $\mathbb{R}$, let $meas(I)$ be the Lebesgue measure of I and denote by $L^1(I)$ the space of Lebesgue integrable functions on I, equipped with the standard norm

$$\|x\|_I = \|x\|_{L^1(I)} = \int_I |x(t)|\, dt.$$

In the case when $I = \mathbb{R}_+$ the norm $\|x\|_{L^1(\mathbb{R}_+)}$ will be briefly denoted by $\|x\|$. Now, let us recall the following criterion of weak noncompactness in the space $L^1(\mathbb{R}_+)$ established by Dieudonne [20]. It will be frequently used in our discussions.

Theorem 2.1. *A bounded set X is relatively weakly compact in $L^1(\mathbb{R}_+)$ if and only if the following two conditions are satisfied:*
(a) for any $\epsilon > 0$ there exists $\delta > 0$ such that if $meas(D) \leq \delta$ then $\int_D |x(t)|dt \leq \epsilon$ for all $x \in X$.
(b) for any $\epsilon > 0$ there exists $\tau > 0$ such that $\int_\tau^\infty |x(t)|dt \leq \epsilon$ for any $x \in X$.

2.2. The Superposition operator

For a fixed interval $I \subset \mathbb{R}$, bounded or not, consider a function $f : I \times \mathbb{R} \to \mathbb{R}$. The function $f = f(t, x)$ is said to satisfy the Carathéodory conditions if it is Lebesgue measurable in t for every fixed $x \in \mathbb{R}$ and continuous in x for almost every $t \in I$. The following theorem due to Scorza Dragoni [21] explains the structure of functions satisfying Carathéodory conditions.