Applied Mathematics and Computation 276 (2016) 417–431

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the asymptotic expansions of the gamma function related to the Nemes, Gosper and Burnside formulas

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ARTICLE INFO

MSC:
33B15
41A60
26D15

Keywords:
Gamma function
Asymptotic expansion
Inequality

ABSTRACT

In this paper, we present new asymptotic expansions of the gamma function related to the Nemes, Gosper and Burnside formulas.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \quad n \in \mathbb{N} := \{1, 2, \dots\} \tag{1.1}$$

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \cdot \sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Eq. (6.1.40)]) by

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right) \tag{1.2}$$

as $x \to \infty$, where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers defined by the following generating function:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi.$$

The following asymptotic formula is due to Laplace

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51\,840x^3} - \frac{571}{2\,488\,320x^4} + \cdots\right) \tag{1.3}$$

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http://dx.doi.org/10.1016/j.amc.2015.12.004

as $x \to \infty$ (see [1, p. 257, Eq. (6.1.37)]). The expression (1.3) is sometimes incorrectly called Stirling's series (see [21, pp. 2–3]). Stirling's formula is in fact the first approximation to the asymptotic formula (1.3). Stirling's formula has attracted much interest of many mathematicians and has motivated a large number of research papers concerning various generalizations and improvements (see [2,4,5,7–11,15,17–20,24,26–36,39–51,54–62] and the references cited therein). See also an overview in [38].

Inspired by (1.1), Burnside [8] found a slightly more accurate approximation than Stirling's formula as follows:

$$n! \sim \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}. \tag{1.4}$$

A much better approximation is the following the Gosper formula [24]:

$$n! \sim \sqrt{2\pi\left(n+\frac{1}{6}\right)}\left(\frac{n}{e}\right)^{n}. \tag{1.5}$$

Ramanujan (see [63, p. 339] and [3, pp. 117–118]) presented the following approximation formula for the gamma function:

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\left(8x^3+4x^2+x+\frac{1}{30}\right)^{1/6}, \quad x\to\infty. \tag{1.6}$$

The Ramanujan formula has been the subject of intense investigations and is reviewed in [6, p. 48, Question 754], and has motivated a large number of research papers (see, for example, [2,9,11,12,17,26–28,34,47,51–54,56]). Karatsuba [28, Eq. (5.5)] developed the approximation formula (1.6) to produce a complete asymptotic expansion:

$$\begin{aligned}\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\Bigg(8x^3+4x^2+x+\frac{1}{30}-\frac{11}{240x}+\frac{79}{3360x^2}+\frac{3539}{201600x^3}-\frac{9511}{403200x^4}\\ -\frac{10051}{716800x^5}+\frac{233934691}{6386688000x^6}+\cdots\Bigg)^{1/6}, \quad x\to\infty\end{aligned} \tag{1.7}$$

(x^{-6} term corrected). Moreover, the author gave a formula for successively determining the coefficients.

Very recently, Chen [14] developed the approximation formula (1.6) to produce various complete asymptotic expansions. For example, Chen [14, Theorem 3] gave a pair of recursive relations for determining the constants λ_ℓ and μ_ℓ such that

$$\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\left(8x^3+4x^2+x+\frac{1}{30}+\sum_{\ell=1}^{\infty}\frac{\lambda_\ell}{(x+\mu_\ell)^{2\ell-1}}\right)^{1/6}, \quad x\to\infty.$$

Namely,

$$\begin{aligned}\Gamma(x+1) \sim \sqrt{\pi}\left(\frac{x}{e}\right)^{x}\Bigg(8x^3+4x^2+x+\frac{1}{30}-\frac{\frac{11}{240}}{x+\frac{79}{154}}+\frac{\frac{459733}{15523200}}{(x+\frac{71181889}{212396646})^3}\\ -\frac{\frac{125134498502528329}{4061997157910630400}}{(x+\frac{59721704420799477948097107}{199336108356217796996884 0050})^5}+\cdots\Bigg)^{1/6}, \quad x\to\infty.\end{aligned} \tag{1.8}$$

From a computational viewpoint, the formula (1.8) improves the Ramanujan–Karatsuba formula (1.7).

Nemes [62, Corollary 4.1] presented the following approximation formula for the gamma function:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^{x}\left(1+\frac{1}{12x^2-\frac{1}{10}}\right)^{x}, \quad x\to\infty. \tag{1.9}$$

The formula (1.9) is stronger than the formula (1.6).

This paper is a continuation of our earlier work [14]. Here we present new asymptotic expansions of the gamma function related to the Nemes, Gosper and Burnside formulas.

2. Lemmas

The following lemmas are required in our present investigation.

Lemma 2.1 (see [16]). *Let*

$$A(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}, \quad x\to\infty$$

be a given asymptotical expansion. Then the composition $\exp(A(x))$ *has asymptotic expansion of the following form*

$$\exp(A(x)) \sim \sum_{n=0}^{\infty} b_n x^{-n}, \quad x\to\infty, \tag{2.1}$$