Applied Mathematics and Computation 276 (2016) 432–441

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A modified iterated projection method adapted to a nonlinear integral equation

Laurence Grammont [a], Paulo B. Vasconcelos [b,*], Mario Ahues [a]

[a] Université de Lyon, Institut Camille Jordan, UMR 5208, 23 rue du Dr Paul Michelon, 42023 Saint-Étienne Cedex 2, France
[b] Faculdade Economia da Universidade Porto and Centro Matemática da Universidade do Porto, Portugal

ARTICLE INFO

MSC:
65J15
45G10
35P05

Keywords:
Nonlinear equations
Newton-like methods
Iterated projection approximation
Integral equations

ABSTRACT

The classical way to tackle a nonlinear Fredholm integral equation of the second kind is to adapt the discretization scheme from the linear case. The Iterated projection method is a popular method since it shows, in most cases, superconvergence and it is easy to implement. The problem is that the accuracy of the approximation is limited by the mesh size discretization. Better approximations can only be achieved for fine discretizations and the size of the linear system to be solved then becomes very large: its dimension grows up with an order proportional to the square of the mesh size. In order to overcome this difficulty, we propose a novel approach to first linearize the nonlinear equation by a Newton-type method and only then to apply the Iterated projection method to each of the linear equations issued from the Newton method. We prove that, for any value (large enough) of the discretization parameter, the approximation tends to the exact solution when the number of Newton iterations tends to infinity, so that we can attain any desired accuracy. Numerical experiments confirm this theoretical result.

1. Introduction

The general framework of this paper is the following. Let $\mathcal{X}$ be a complex Banach space and $F : \mathcal{O} \subseteq \mathcal{X} \to \mathcal{X}$ a nonlinear Fréchet differentiable operator defined on a nonempty open set $\mathcal{O}$ of $\mathcal{X}$. The problem is set as

$$\text{Find } \varphi \in \mathcal{O} : \quad F(\varphi) = 0, \tag{1}$$

where 0 is the null vector of $\mathcal{X}$.

Our aim is to treat a special case of Eq. (1): a nonlinear Fredholm integral equation of the second kind:

$$\text{Find } \varphi \in \mathcal{O} : \quad \varphi - K(\varphi) = f, \tag{2}$$

for a given function $f \in \mathcal{X}$, where

$$K(x)(s) := \int_0^1 \kappa(s, t, x(t))\, dt, \quad x \in \mathcal{O},\ s \in [0, 1],$$

and the kernel κ is a real-valued function of three variables :

$$(s, t, u) \in [0, 1]\times[0, 1]\times\mathbb{R} \mapsto \kappa(s, t, u) \in \mathbb{R},$$

with enough regularity so that K is twice Fréchet-differentiable on $\mathcal{O}$.

* Corresponding author. Tel.: +351 22 0462374; fax: +351 22 5505050.
E-mail addresses: laurence.grammont@univ-st-etienne.fr (L. Grammont), pjv@fep.up.pt (P.B. Vasconcelos), mario.ahues@univ-st-etienne.fr (M. Ahues).

http://dx.doi.org/10.1016/j.amc.2015.12.019
0096-3003/© 2016 Published by Elsevier Inc.

Let $T := K'$ denotes the Fréchet derivative of K, i.e., for all $x \in \mathcal{O}$,

$$T(x)h(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,x(t))h(t)\,dt, \quad h \in \mathcal{X}, s \in [0,1]. \tag{3}$$

In this context, the usual way to build a numerical approximation to the solution of (1) is to discretize it and obtain a nonlinear system of equations in a finite dimensional space, then to apply the Newton method to the discrete nonlinear problem, and to solve the corresponding finite dimensional linear problem at each iteration.

Concerning the integral equations of the form (2), there are mainly two types of discretizations : Nyström-type methods, based on numerical quadrature formulae, and projection methods. The approximate solution ψ_n of a projection method is the solution of the approximate equation:

$$\text{find } \psi_n \in \mathcal{O}: \quad \psi_n - K_n(\psi_n) = f_n, \tag{4}$$

where K_n is an approximation of the operator K depending on a projection operator π_n onto a finite dimensional space $\mathcal{X}_n$ and f_n an approximation of f. For the simplest projection method, Galerkin, if π_n is the orthogonal projection, and collocation, if π_n is an interpolatory projection, $\psi_n \in \mathcal{X}_n$, $K_n = \pi_n K \pi_n$ and $f_n = \pi_n f$. For the Kantorovich projection method $\psi_n \in \mathcal{X}$, $K_n = \pi_n K$ and $f_n = f$. For the Iterated projection method $\psi_n \in \mathcal{X}$, $K_n = K\pi_n$ and $f_n = f$. When K is a linear operator, the latter method is also called Sloan method (see [13]). This method is particularly interesting because it is quite easy to implement and its order of convergence can be higher than that of the Galerkin or the collocation methods (see [5,6,8]). General features about that strategy– discretization then linearization – can be found in the survey [7] by Atkinson and Flores. That is the usual way to treat (2).

The problem with this way of thinking is that the accuracy of the approximate solution is limited by n, the parameter of the discretization which also decides the size of the linear system to be solved. Whatever the accuracy of the finite dimensional Newton's method can be, the discretization error remains.

Our gamble in the general framework is the following: if one linearizes the functional Eq. (2) first, via the Newton method, one is led to a sequence of functional linear equations to be discretized in order to solve a finite-dimensional linear problem. Can we attain any desired accuracy for a fixed discretization parameter n by applying more Newton iterations? For the Galerkin or collocation method, it is equivalent to begin with linearization or discretization. But the bet is won when the discretization process is the Nyström method or the Kantorovich projection method (see [10,11]). In both cases the approximation issued from linearizing first (Strategy L) is shown to converge to the exact solution φ of (1) when the number of Newton iterations tends to infinity, while, when we discretize first (Strategy D) by Nyström or Kantorovich projection methods, the Newton iterates converge to the solution of the approximate equation. Notice that, in general, Strategy D is easier to implement and shows a faster convergence. Roughly speaking, we have the choice of converging slowly to the exact solution with Strategy L or converging fast but to a poor solution with Strategy D.

In papers [10,11], the framework is such that we have the norm convergence of the approximate operators K_n to K, and also $\|K_n' - K'\| \to 0$. For the Iterated projection approximate operator $K_n = K\pi_n$, we have only collectively compact convergence of the Fréchet derivative of K_n to the Fréchet derivative of K. Hence the situation is more difficult to handle.

The aim of this paper is to study the behavior of Strategy L, compared to the classical Strategy D when the discretization process is the Iterated projection method.

In a general framework, Strategy L can be considered as the application of Newton's method leading to a sequence $(\varphi^{(k)})_{k \geq 0}$ defined by:

$$F'(\varphi^{(k)})(\varphi^{(k+1)} - \varphi^{(k)}) = -F(\varphi^{(k)}), \quad \varphi^{(0)} \in \mathcal{O}, \tag{5}$$

where F' denotes the Fréchet derivative of F. For convergence results on Newton and Newton-like methods, see the book of Argyros [3] and the papers [1,4,9].

If $F'(\varphi^{(k)})$ is invertible, Eq. (5) can be rewritten as

$$\varphi^{(k+1)} = \varphi^{(k)} - F'(\varphi^{(k)})^{-1} F(\varphi^{(k)}).$$

Then any discretization of this equation can be written as

$$\varphi_n^{(0)} \in \mathcal{V} \subset \mathcal{O}, \quad \varphi_n^{(k+1)} = \varphi_n^{(k)} - \Sigma_n(\varphi_n^{(k)}) F(\varphi_n^{(k)}), \tag{6}$$

where $\mathcal{V}$ is an open subset of $\mathcal{O}$ and $\Sigma_n : \mathcal{V} \subset \mathcal{O} \to \mathcal{L}(\mathcal{X})$ is such that, in some discretized sense, for each $x \in \mathcal{V}$, $\Sigma_n(x)$ is an approximation of $F'(x)^{-1}$, where $\mathcal{L}(\mathcal{X})$ denotes the space of bounded linear operators from $\mathcal{X}$ into itself. Note that the discretization process is described by the operator Σ_n.

In Section 2, we provide conditions on the operator F and on the discretization process represented by Σ_n that guarantee the convergence of the sequence $(\varphi_n^{(k)})_{k \in \mathbb{N}}$ of solutions of Eq. (6) to the exact solution φ of Eq. (1).

In Section 3, we use this result to suggest a new way of implementing the Iterated projection method to approximate the solution for a nonlinear Fredholm equation of the second kind. We prove that the new Iterated projection discretization scheme fulfills the conditions required to apply the general result of Section 2.

Section 4 is devoted to a numerical comparison between Strategy D and Strategy L. It confirms the theoretical analysis.