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On blow-up criteria for a new Hall-MHD system

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ABSTRACT

This paper investigates a new Hall-MHD system in $\mathbb{R}^3$. Besides local well-posedness for strong solutions and global existences for weak solutions, some blow-up criteria are established.

1. Introduction

In this paper, we consider the following Hall-MHD system [1–4]:

$$\operatorname{div} u = \operatorname{div} b = 0, \tag{1.1}$$

$$\partial_t u + u \cdot \nabla u + \nabla\left(\pi + \frac{1}{2}|b|^2\right) - \Delta u = b \cdot \nabla b, \tag{1.2}$$

$$\partial_t b - \left(\frac{\delta_e}{L_0}\right)^2 \partial_t \Delta b + u \cdot \nabla b - b \cdot \nabla u - \Delta b = \frac{\delta_i}{L_0}\frac{1}{\rho}\operatorname{rot}(b \times \operatorname{rot} b) - \left(\frac{\delta_e}{L_0}\right)^2 \frac{1}{\rho}\operatorname{rot}((u \cdot \nabla)\operatorname{rot} b), \tag{1.3}$$

$$(u, b)(\cdot, 0) = (u_0, b_0)(\cdot) \ \text{ in } \ \mathbb{R}^3. \tag{1.4}$$

Here u is the fluid velocity field, π is the pressure and b is the magnetic field. L_0 is the normalizing length limit, δ_e and δ_i denote electron and ion inertia respectively, and ρ is the fluid density. For simplicity, we will take $\delta_e = \delta_i = L_0 = \rho = 1$.

The applications of the Hall-MHD system cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo.

When $\delta_e = 0$, the system (1.1)–(1.3) reduces to the incompressible Hall-MHD equations, which has been received many studies [5–14]. The paper [5] gave a derivation of (1.1)–(1.3) from a two-fluid Euler–Maxwell system. Chae et al. [8] proved the local existence of smooth solutions. Chae and Lee [6] and Fan and Ozawa [11] proved some regularity criteria. For the compressible and density-dependent Hall-MHD, we refer to [15] and [16].

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When the last term in (1.3) is neglected, the system (1.1)–(1.3) reduced to the two-fluid MHD equation. Chen and Miao [17] proved the local existence of strong solutions and some regularity criteria.

The aim of this paper is to investigate well-posedness and blow-up criteria for (1.1)–(1.4). First, we have the following local well-posedness theorem for the strong solution.

Theorem 1.1. *Let $u_0 \in H^s, b_0 \in H^{s+1}$ with $s \geq 3$ and $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Then there exist $T > 0$ and a unique strong solution (u, b) to the problem (1.1)–(1.4) satisfying*

$$u \in L^\infty(0, T; H^s) \cap L^2(0, T; H^{s+1}), \ \ b \in L^\infty(0, T; H^{s+1}). \tag{1.5}$$

Since the proof of Theorem 1.1 is very similar to that in [17], we omit the details here. At the same time, it is easy to establish a global existence for weak solutions similarly to that for the standard MHD equations.

Theorem 1.2. *Let $u_0 \in L^2, b_0 \in H^1$ and $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Then the problem has a weak solution u, b satisfying energy inequality for any $T > 0$.*

Note that the last term in (1.3) can be rewritten as $-\sum_{i=1}^3 \operatorname{rot}(\partial_i(u_i \operatorname{rot} b))$, but we omit the detailed proof here.

The following blow-up criteria are the main results and will be proved in Section 2.

Theorem 1.3. *Let $u_0 \in H^s, b_0 \in H^{s+1}$ with $s \geq 3$ and $\operatorname{div} u_0 = \operatorname{div} b_0 = 0$. Let (u, b) be the unique solution constructed in Theorem 1.1. If one of the following two conditions*

$$(i) \quad \int_0^T (\|\nabla u(t)\|_{L^\infty} + \|b(t)\|_{BMO}^2) dt < \infty, \tag{1.6}$$

$$(ii) \quad \int_0^T (\|\nabla u(t)\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla b(t)\|_{\dot{B}^0_{\infty,\infty}}) dt < \infty, \tag{1.7}$$

holds for some $0 < T < \infty$, then the solution (u, b) can be extended beyond $T > 0$.

Here $\dot{B}^0_{\infty,\infty}$ denotes the homogeneous Besov space and BMO is the space of bounded mean oscillation.

Definition 1.1. (see [18]). Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity that satisfies $\hat{\phi} \in C_0^\infty(B_2 \setminus B_{1/2})$, $\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j}\xi)$ and $\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1$ for any $\xi \neq 0$, where $\hat{\phi}$ is the Fourier transform of ϕ and B_r is the ball with radius r centered at the origin. The homogeneous Besov space is defined as

$$\dot{B}^s_{p,q} := \left\{ f \in \mathcal{S}'/\mathcal{P} : \|f\|_{\dot{B}^s_{p,q}} < \infty \right\}$$

with the norm

$$\|f\|_{\dot{B}^s_{p,q}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} \|2^{js}\Delta_j f\|_{L^p}^q\right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \sup_{j\in\mathbb{Z}} \|2^{js}\Delta_j f\|_{L^p}, & q = \infty, \end{cases}$$

for all $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ and $\Delta_j f := \phi_j * f$, where $\mathcal{S}'$ is the space of tempered distributions and $\mathcal{P}$ is the space of polynomials.

We will also use the following logarithmic Sobolev inequalities [19,20]:

$$\|\nabla u\|_{L^\infty} \leq C(1 + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} \log(e + \|u\|_{H^s})), \tag{1.8}$$

$$\|b\|_{L^\infty} \leq C(1 + \|b\|_{BMO} \log^{1/2}(e + \|b\|_{H^2})), \tag{1.9}$$

with $s > \frac{5}{2}$.

2. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3. We only need to establish a priori estimates.

First, we can do the following energy estimates.

Testing (1.2) by u and using (1.1), we see that

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 dx + \int |\nabla u|^2 dx = \int (b \cdot \nabla) b \cdot u dx. \tag{2.1}$$

Testing (1.3) by b and using (1.1), we infer that

$$\frac{1}{2}\frac{d}{dt}\int (|b|^2 + |\nabla b|^2) dx + \int |\nabla b|^2 dx = \int (b \cdot \nabla) u \cdot b dx. \tag{2.2}$$