Applied Mathematics and Computation 274 (2016) 41–46

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# The generalized 3-connectivity of star graphs and bubble-sort graphs

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ARTICLE INFO

*MSC:*
05C40
90C35

*Keywords:*
Generalized 3-connectivity
Internally disjoint trees
Cayley graphs
Star graphs
Bubble-sort graphs

ABSTRACT

For $S \subseteq G$, let $\kappa(S)$ denote the maximum number $r$ of edge-disjoint trees $T_1, T_2, \ldots, T_r$ in $G$ such that $V(T_i) \cap V(T_j) = S$ for any $i, j \in \{1, 2, \ldots, r\}$ and $i \neq j$. For every $2 \leq k \leq n$, the *generalized k-connectivity* of $G$ $\kappa_k(G)$ is defined as the minimum $\kappa(S)$ over all $k$-subsets $S$ of vertices, i.e., $\kappa_k(G) = \min\{\kappa(S) | S \subseteq V(G) \text{ and } |S| = k\}$. Clearly, $\kappa_2(G)$ corresponds to the traditional connectivity of $G$. The generalized $k$-connectivity can serve for measuring the capability of a network $G$ to connect any $k$ vertices in $G$. Cayley graphs have been used extensively to design interconnection networks. In this paper, we restrict our attention to two classes of Cayley graphs, the star graphs $S_n$ and the bubble-sort graphs $B_n$, and investigate the generalized 3-connectivity of $S_n$ and $B_n$. We show that $\kappa_3(S_n) = n - 2$ and $\kappa_3(B_n) = n - 2$.



## 1. Introduction

The traditional *connectivity* $\kappa(G)$ of a graph $G$ is defined as the minimum cardinality of a subset $Q$ of vertices of $G$ such that $G - Q$ is disconnected or trivial. A graph $G$ is said to be *k-connected* if $\kappa(G) \geq k$. Two distinct paths are *internally disjoint* if they have no internal vertices in common. A well-known theorem of Whitney [20] provides an equivalent definition of connectivity. For each 2-subset $S = \{u, v\}$ of vertices of $G$, let $\kappa(S)$ denote the maximum number of internally disjoint $(u, v)$-paths in $G$. Then $\kappa(G) = \min\{\kappa(S) | S \subseteq V \text{ and } |S| = 2\}$.

As a means of strengthening the connectivity, the generalized connectivity was introduced, among the same definition given by other authors, by Hager [3,4]. Let $G$ be a nontrivial connected graph of order $n$. For $S \subseteq V(G)$, $T_1$ and $T_2$ are two *internally disjoint trees connecting* $S$ if $T_1$ and $T_2$ are edge-disjoint and $V(T_1) \cap V(T_2) = S$ (note that the two trees are vertex-disjoint in $G - S$). Let $\kappa(S)$ denote the maximum number of internally disjoint trees connecting $S$ in $G$. The *generalizedk-connectivity* of $G$, denoted by $\kappa_k(G)$, is then defined by $\kappa_k(G) = \min\{\kappa(S) | S \subseteq V(G) \text{ and } |S| = k\}$, where $2 \leq k \leq n$. Thus, when $k = 2$, the generalized 2-connectivity $\kappa_2(G)$ of $G$ is exactly the connectivity $\kappa(G)$, namely $\kappa_2(G) = \kappa(G)$. There have been many results on the generalized connectivity, see [8–14] and a survey [15]. The concept of generalized connectivity is related to another generalization of traditional connectivity, called *rainbow connection number*. Let $G$ be a nontrivial connected graph on which an edge-coloring $c : E(G) \to \{1, 2, \ldots, n\}$, is defined, where adjacent edges may be colored the same. A path is rainbow if no two edges of it are colored the same. An edge-coloring graph $G$ is rainbow connected if any two vertices are connected by a rainbow path. We define the rainbow connection number of a connected graph $G$, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make $G$ rainbow connected. The rainbow connection number has been widely studied [6,16–18].

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http://dx.doi.org/10.1016/j.amc.2015.11.016
0096-3003/

The underlying topology of a computer interconnection network can be modeled by a graph $G$, and the connectivity $\kappa(G)$ of $G$ is an important measure for fault tolerance of the network. In general, the larger $\kappa(G)$ is, the more reliable the network is. However, if one wants to know how tough a network can be, for the connection of a set of vertices, then the generalized $k$-connectivity can serve for measuring the capability of a network $G$ to connect any $k$ vertices in $G$.

Since Cayley graphs have been used extensively to design interconnection networks, the study of the generalized $k$-connectivity of Cayley graphs is very significative.

Let $X$ be a group and $S$ be a subset of $X$. The *Cayley digraph* $Cay(X, S)$ is a digraph with vertex set $X$ and arc set $\{(g, gs)|g \in X, s \in S\}$. Clearly, if $S = S^{-1}$, where $S^{-1} = \{s^{-1}|s \in S\}$, then $Cay(X, S)$ can be made into an undirected graph. Cayley (di)graphs have a lot of properties which are desirable in an interconnection network [5,7]: vertex symmetry makes it possible to use the same routing protocols and communication schemes at all nodes; hierarchical structure facilitates recursive constructions; high fault tolerance implies robustness, among others.

Now, we consider Cayley graphs $Cay(X, S)$ when the group $X$ is a permutation group. Denote by $Sym(n)$ the group of all permutations on $\{1, \ldots, n\}$. Let $(p_1 p_2 \ldots p_n)$ denote a permutation on $\{1, \ldots, n\}$ and $(ij)$, which is called a *transposition*, denote the permutation that swaps the objects at positions $i$ and $j$ (not swapping element $i$ and $j$), that is, $(p_1 \ldots p_i \ldots p_j \ldots p_n)(ij) = (p_1 \ldots p_j \ldots p_i \ldots p_n)$. Let $\mathcal{T}$ be a set of transpositions and $G(\mathcal{T})$ be the graph on $n$ vertices $\{1, 2, \ldots, n\}$ such that there is an edge $ij$ in $G(\mathcal{T})$ if and only if the transposition $(ij) \in \mathcal{T}$. The graph $G(\mathcal{T})$ is called the *transposition generating graph* of $Cay(Sym(n), \mathcal{T})$.

Moreover, if $G(\mathcal{T})$ is a tree, we call $G(\mathcal{T})$ a *transposition tree* and denote $Cay(Sym(n), \mathcal{T})$ by $\Gamma_n$. Specially, if $G(\mathcal{T}) \cong K_{1,n-1}$, then $Cay(Sym(n), \mathcal{T})$ is called a *star graph* $S_n$; and $Cay(Sym(n), \mathcal{T})$ is called a *bubble-sort graph* $B_n$ if $G(\mathcal{T}) \cong P_n$.

In this paper, we study the generalized 3-connectivity of the star graph $S_n$ and the bubble-sort graph $B_n$, and show that $\kappa_3(S_n) = n - 2$ and $\kappa_3(B_n) = n - 2$.

## 2. Preliminaries

We first introduce some notation and results that will be used throughout the paper.

We consider finite and simple graphs $G$. $V(G)$ and $E(G)$ denote its vertex set and its edge set respectively. For $v \in V(G)$, denote by $N_G(v)$ the set of neighbors of $v$ in $G$. For a subset $U \subseteq V(G)$, let $N(U) := (\cup_{u \in U} N(u)) \backslash U$, and the subgraph induced by $U$ is denoted by $G[U]$. Sometimes, we use a graph itself to represent its vertex set, for instance, $N(G_1)$ means $N(V(G_1))$, where $G_1$ is a subgraph of $G$.

**Lemma 2.1** ([14])**.** *Let G be a connected graph with minimum degree* $\delta$. *Then* $\kappa_3(G) \leq \delta$. *In particular, if there are two adjacent vertices of degree* $\delta$, *then* $\kappa_3(G) \leq \delta - 1$.

**Lemma 2.2** ([14])**.** *Let G be a connected graph with n vertices. For every two integers k and r with* $k \geq 0$ *and* $r \in \{0, 1, 2, 3\}$, *if* $\kappa(G) = 4k + r$, *then* $\kappa_3(G) \geq 3k + \lceil \frac{r}{2} \rceil$. *Moreover, the lower bound is sharp.*

**Lemma 2.3** (The Fan Lemma [1], p. 214)**.** *Let G be a k-connected graph, x a vertex of G, and let* $Y \subseteq V - \{x\}$ *be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y, namely there exists a family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

Recall that $\Gamma_n = Cay(Sym(n), \mathcal{T})$ represents the Cayley graphs generated by transposition trees $G(\mathcal{T})$. The Cayley graphs $\Gamma_n$ are $(n-1)$-regular bipartite graphs and have $n!$ vertices, see [7] for the details.

Without loss of generality, we assume that for the star graph $S_n$, the transposition generating graph is $G(\mathcal{T}) = \{\{1, \ldots, n\}, \{12, 13, \ldots, 1n\}\}$ and for the bubble-sort graph $B_n$, the transposition generating graph is $G(\mathcal{T}) = \{\{1, \ldots, n\}, \{12, 23, \ldots, (n-1)n\}\}$ throughout the paper.

Now we give some useful properties, which can be found in [2,19,21].

**Lemma 2.4** ([2,21])**.** $\kappa(\Gamma_n) = n - 1$.

Thus, $\kappa(S_n) = n - 1$ and $\kappa(B_n) = n - 1$.

**Property 2.1.** [19] For $\Gamma_n$, if $n$ is a leaf of $G(\mathcal{T})$, then $\Gamma_n$ can be decomposed into $n$ disjoint copies of $\Gamma_{n-1}$, say $\Gamma^1_{n-1}, \Gamma^2_{n-1}, \ldots, \Gamma^n_{n-1}$, where $\Gamma^i_{n-1}$ is an induced subgraph by vertex set $\{(p_1 p_2 \ldots p_{n-1} i)|(p_1 \ldots p_{n-1})$ ranges over all permutations of $\{1, \ldots, n\} \backslash \{i\}\}$. We denote this decomposition by $\Gamma_n = \Gamma^1_{n-1} \oplus \Gamma^2_{n-1} \oplus \ldots \oplus \Gamma^n_{n-1}$.

Thus, by Property 2.1 $S_n = S^1_{n-1} \oplus S^2_{n-1} \oplus \cdots \oplus S^n_{n-1}$ and $B_n = B^1_{n-1} \oplus B^2_{n-1} \oplus \cdots \oplus B^n_{n-1}$

**Property 2.2.** [2] Consider the Gayley graphs $\Gamma_n$. Let $(tn) \in \mathcal{T}$ be a pendant edge of $G(\mathcal{T})$. For any vertex $u$ of $\Gamma^i_{n-1}$, $u(tn)$ is the unique neighbor of $u$ outside of $\Gamma^i_{n-1}$, is called the *out-neighbor* of $u$, written $u'$. We call the neighbors of $u$ in $\Gamma^i_{n-1}$ the *in-neighbors* of $u$. Any two distinct vertices of $\Gamma^i_{n-1}$ have different out-neighbors. Hence, there are exactly $(n-2)!$ independent edges between $\Gamma^i_{n-1}$ and $\Gamma^j_{n-1}$ if $i \neq j$, that is, $|N(\Gamma^i_{n-1}) \cap V(\Gamma^j_{n-1})| = (n-2)!$ if $i \neq j$.

We give the following result.